Primes of height one and a class of Noetherian finitely presented algebras*

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Abstract

Constructions are given of Noetherian maximal orders that are finitely presented algebras over a field K, defined by monomial relations. In order to do this, it is shown that the underlying homogeneous information determines the algebraic structure of the algebra. So, it is natural to consider such algebras as semigroup algebras $K[S]$ and to investigate the structure of the monoid S. The relationship between the prime ideals of the algebra and those of the monoid S is one of the main tools. Results analogous to fundamental facts known for the prime spectrum of algebras graded by a finite group are obtained. This is then applied to characterize a large class of prime Noetherian maximal orders that satisfy a polynomial identity, based on a special class of submonoids of polycyclic-by-finite groups. The main results are illustrated with new constructions of concrete classes of finitely presented algebras of this type.

Because of the role of Noetherian algebras in the algebraic approach in noncommutative geometry, new concrete classes of finitely presented algebras recently gained a lot of interest. Via the applications to the solutions of the Yang-Baxter equation, one such class also widens the interest into other fields, such as mathematical physics. These algebras are finitely generated, say by elements $x_1, \ldots, x_n$, and they have a presentation defined by $\binom{n}{2}$ monomial relations of the form $x_i x_j = x_k x_l$ so that every word $x_i x_j$ appears at most once in a relation. Such algebras have been extensively studied, for example in [11, 12, 14, 20, 33]. It was shown that they have a rich algebraic structure that resembles that of a polynomial algebra in finitely many commuting generators; in particular, they are prime Noetherian maximal orders. Clearly, these algebras can be considered as semigroup algebras $K[S]$, where S is a monoid defined via a presentation as that of the algebra. It turns out that these algebras are closely related to group algebras since S is a submonoid of a torsion-free abelian-by-finite group.

In this paper we further explore new constructions of Noetherian maximal orders that are finitely presented algebras which are defined via monomial relations. In other words, we look for new constructions of semigroup algebras $K[S]$ of this type. In general, it remain unsolved problems to characterize when an arbitrary semigroup algebra is Noetherian and when it is a prime Noetherian maximal order. Recall that the former question even for group algebras has been unresolved. The only class of groups for which a positive answer has been given is that of the polycyclic-by-finite groups. Hence it is natural to consider the problems first for semigroup algebras $K[S]$ of submonoids S of polycyclic-by-finite groups. A structural characterization of such algebras $K[S]$ that are right Noetherian was obtained by the authors in [22, 23]. Namely, S has a group of quotients $G = SS^{-1}$ in which there is a normal subgroup H of finite index so that $[H, H] \subseteq S$ and $S \cap H$ is finitely generated, or equivalently, S has a group of quotients G that contains normal subgroups F and N so that $F \subseteq N$, F is a subgroup of finite index in the unit group $U(S)$ of S, the group N/F is abelian, G/N is finite and $S \cap N$ is finitely generated. It follows that $K[S]$ is right Noetherian if and only if $K[S]$ is left Noetherian. We simply say that $K[S]$ is Noetherian. Also, if S is a submonoid of a finitely generated group G that has an abelian subgroup A of finite index, then $K[S]$ is Noetherian if and only if $S \cap A$ is finitely generated. Recall that if $K[S]$ is Noetherian then S is finitely generated and $K[S \cap G_1]$ is Noetherian for every subgroup G_1 of G. If, furthermore, G_1 is of finite index in G then $K[S]$ is a finitely generated right (and left) $K[S \cap G_1]$-module. Due to a result of Chin and Quinn [8] on rings graded by polycyclic-by-finite groups, $K[S]$ is Noetherian if and only if S satisfies the ascending chain condition on right (equivalently, left) ideals.

Concerning the second problem. If S is an abelian monoid, Anderson [1, 2] proved that $K[S]$ is a prime Noetherian maximal order if and only if S is a submonoid of a finitely generated torsion free abelian group, S satisfies the ascending chain condition on ideals and S is a maximal order in its group of quotients. More generally, Chouinard [9] proved that a commutative monoid algebra $K[S]$ is a Krull domain if and only if S is a submonoid of a torsion free abelian group which satisfies the ascending chain condition on cyclic subgroups and S is a Krull order in its group of quotients. In particular, it turns out that the height one prime ideals of $K[S]$ determined by the minimal primes of S are crucial. Brown characterized group algebras $K[G]$ of a polycyclic-by-finite group G that are Noetherian prime maximal orders [3, 4]. This turns out to be always the case if G is a finitely generated torsion free abelian-by-finite group (equivalently, $K[G]$ is a Noetherian domain that satisfies a polynomial identity, PI for short). In this situation all height one primes are principally generated by a normal element. So, in the terminology of Chatters and Jordan [7], $K[G]$ is a unique factorization ring. The authors described in [21] when a semigroup algebra of a submonoid S of a finitely generated abelian-by-finite group is a Noetherian maximal order that is a domain. The description is fully in terms of the monoid S. Examples of such monoids are the binomial monoids and more general monoids of I-type. As mentioned earlier in the introduction, the latter were introduced by Gateva-Ivanova and Van den Bergh in [14] and studied in

several papers (see for example [11, 12, 15, 20, 33]) and generalized to monoids of IG-type in [18].

In this paper we continue the investigations of prime Noetherian maximal orders $K[S]$ for submonoids S of a polycyclic-by-finite group $G = SS^{-1}$. Of course, knowledge of prime ideals is fundamental. The primes not intersecting S come from primes in the group ring $K[SS^{-1}]$. These are rather well understood through the work of Roseblade (see for example [31]), and in particular, the height one primes can be handled via Brown's result. The crucial point in our investigations are thus the primes of $K[S]$ intersecting S non-trivially. In case $K[S]$ is Noetherian and $G = SS^{-1}$ is torsion free, the following information was proved in [22].

1. If Q is a prime ideal of S then $K[Q]$ is a prime ideal of $K[S]$.

2. The height one prime ideals of $K[S]$ intersecting S non-trivially are precisely the ideals $K[Q]$ with Q a minimal prime ideal of S.

Note that G being torsion free is equivalent with $K[G]$ being a domain (see [32, Theorem 37.5]).

In the first part of this paper we continue the study of prime ideals in case $K[S]$ is prime (and thus G is not necessarily torsion free). We show that the first property does not remain valid, however the second part on primes of height one still remains true. As a consequence, we establish going up and going down properties between prime ideals of S and prime ideals of $S \cap H$, where H is a subgroup of finite index in G. These are the analogs of the important results (see [32, Theorem 17.9]) known on the prime ideal behaviour between a ring graded by a finite group and its homogeneous component of degree e (the identity of the grading group). As an application it is shown that the classical Krull dimension clKdim$(K[S])$ of $K[S]$ is the sum of the prime dimension of S and the plinth length of the unit group U(S). Also, a result of Schelter is extended to the monoid S: the prime dimension of S is the sum of the height and depth of any prime ideal of S.

The information obtained on primes of height one then allows us in the second section to determine when a semigroup algebra $K[S]$ is a prime Noetherian maximal order provided that $G = SS^{-1}$ is a finitely generated abelian-by-finite group (that is, $K[S]$ satisfies a polynomial identity). The result reduces the problem to the structure of the monoid S (in particular S has to be a maximal order within its group of quotients G) and to that of G. The characterization obtained generalizes the one given in [21] in case G is torsion free and it shows that the action of G on minimal primes of some abelian submonoid of S is very important (as was also discovered for the special case where S is a finitely generated monoid of IG-type in [18]). Finally, in the last section, we prove a useful criterion for verifying the maximal order property of such S. We then show how this, together with our main result, can be used to build new examples of finitely presented algebras R (defined by monomial relations) that are maximal orders. Since the main result of the paper deals with semigroup algebras $K[S]$ of submonoids S of groups, we first have to check that R is defined by such

submonoids. So, in particular, in the third section we show how to go from the language of presentations of algebras to the language of monoids and their semigroup algebras.

1 Prime spectrum

Let K be a field. Recall that if S is a monoid with a group of quotients $G = SS^{-1}$ then the semigroup algebra $K[S]$ is prime if and only if $K[G]$ is prime ([28, Theorem 7.19]), or equivalently G does not contain nontrivial finite normal subgroups. The latter can also be stated as $\Delta^+(G) = \{1\}$, where $\Delta^+(G)$ is the characteristic subgroup of G consisting of the periodic elements with finitely many conjugates (see [32, Theorem 5.5]). By $\Delta(G)$ one denotes the subgroup of G consisting of the elements with finitely many conjugates.

Also recall that an equivalence relation ρ on a semigroup S is said to be a congruence if $s\rho t$ implies $su\rho tu$ and $us\rho ut$ for every $u \in S$. The set of ρ-classes, denoted S/ρ, has a natural semigroup structure inherited from S. In case S has a group of quotients G and H is a normal subgroup of G we denote by $\sim_H$ the congruence on S defined by: $s \sim_H t$ if and only if $s = ht$ for some $h \in H$. If G is a polycyclic-by-finite group then by $\mathrm{h}(G)$ we mean the Hirsch rank of G.

We often will make use of the following fact ([29, Lemma 3.1]). If S is a submonoid of a finitely generated abelian-by-finite group G then S has a group of quotients $S\mathrm{Z}(S)^{-1} = \{sz^{-1} \mid s \in S,\ z \in \mathrm{Z}(S)\}$. It follows that if T is a submonoid of a group G that contains a normal subgroup F so that $F \subseteq T$ and G/F is finitely generated and abelian-by-finite (for example $T \subseteq G$, G is polycyclic-by-finite and T satisfies the ascending chain condition on right ideals, see [23]) then T has a group of quotients that is obtained by inverting the elements u of T that are central modulo F. In particular, every (nonempty) ideal of T contains such an element u.

Theorem 1.1 *Let S be a submonoid of a polycyclic-by-finite group, say with a group of quotients G, and let K be a field. Assume that S satisfies the ascending chain condition on right ideals and G does not contain nontrivial finite normal subgroups. Then the height one prime ideals P of $K[S]$ such that $P \cap S \neq \emptyset$ are exactly the ideals of the form $P = K[Q]$, where Q is a minimal prime ideal of S.*

Proof. Let P be a height one prime ideal of $K[S]$ such that $Q = S \cap P \neq \emptyset$. By [30] (or see [24, Theorem 4.5.2], and also [24, Corollary 4.5.7]), S/Q embeds into $\mathcal{M}_n(H)$, the semigroup of $n \times n$ monomial matrices over a group $H = TT^{-1}$ for a subsemigroup T of S such that $T \cap Q = \emptyset$. Furthermore, $K[S]/K[Q]$ embeds in the matrix algebra $M_n(K[H])$ and the latter is a localization of $K[S]/K[Q]$ with respect to an Ore set that does not intersect $P/K[Q]$. Subsemigroups of S not intersecting Q can be identified with subsemigroups of $\mathcal{M}_n(H)$. It is also known that there exists an ideal I of S containing Q such that the nonzero elements of I/Q are the matrices in $S/Q \subseteq \mathcal{M}_n(H)$ with exactly one nonzero entry. Furthermore, T may be chosen so that $T \subseteq I \setminus Q$ and actually T may be

identified with $(e_{11}\mathcal{M}_n(H)e_{11} \cap (S/Q)) \setminus \{0\}$, where e_{11} is a diagonal idempotent of rank one in $\mathcal{M}_n(H)$. Clearly I/Q is an essential ideal in S/Q. We know that S contains a normal subgroup F of G such that G/F is abelian-by-finite. Then $H \subseteq FH$ and clearly $FH \cap Q = \emptyset$ and $FH = FT(FT)^{-1}$. Clearly $FTTF \subseteq FHF = FH$, whence $(FT)(TF) \cap Q = \emptyset$. Since T is a diagonal component of I and $FT, TF \subseteq I$, by the matrix pattern on the matrices of rank one in $\mathcal{M}_n(H)$ it follows that $TF = FT = T$. Hence $H = \mathrm{gr}(T) = \mathrm{gr}(TF) = HF$.

Notice that we have a natural homomorphism $\phi : K[S]/K[Q] \longrightarrow K[S]/P$. Since $K[S]$ is Noetherian and as $M_n(K[H])$ is a localization of $K[S]/K[Q]$ with respect to an Ore set of regular elements that does not intersect $P/K[Q]$, there exists a prime ideal R in $K[H]$ such that $M_n(K[H]/R)$ is a localization of $K[S]/P$, see [17, Theorem 9.22]. Moreover, since $P/K[Q]$ is a prime ideal of $K[S]/K[Q]$, we also have that $M_n(K[H]/R) \subseteq M_n(M_t(D))$ for a division ring D and a positive integer t such that $M_t(D)$ is the ring of quotients of $K[H]/R$ and ϕ extends to a homomorphism $\phi' : M_n(K[H]) \longrightarrow M_n(K[H]/R)$. Let ρ_P be the congruence on S defined by the condition: $(x, y) \in \rho_P$ if $x - y \in P$. Since the image of a nonzero entry of a matrix $s \in \mathcal{M}_n(H)$ under ϕ' is invertible in $K[H]/R$, the rank of the matrix $\phi'(s)$ is equal to t multiplied by the number of nonzero entries of s. Thus, the ideal J of matrices of rank at most t in $S/\rho_P \subseteq M_{nt}(D)$ satisfies $\phi^{-1}(J) = I$. Moreover, $S/\rho_P = \phi'(S/Q) \subseteq \mathcal{M}_n(GL_t(D))$ inherits the monomial pattern of $S/Q \subseteq \mathcal{M}_n(H)$. From [22, Corollary 4.4] we know that there is a right Ore subsemigroup U of I such that $\mathrm{h}(UU^{-1}) = \mathrm{h}(G) - 1$ and $U \cap P = \emptyset$, and also the image U' of U in S/ρ_P is contained in a diagonal component of J (viewed as a monomial semigroup over $GL_t(D)$). Then U can be identified with a subsemigroup of S/Q and it is contained in a diagonal component of S/Q. It thus follows that $\mathrm{h}(H) \geq \mathrm{h}(\mathrm{gr}(U)) = \mathrm{h}(G) - 1$.

By the remark before the theorem we know that every ideal of S contains an element of S that is central modulo F. So, choose $z \in Q$ such that $zx \in xzF$ for every $x \in S$. Suppose that $z^m \in H$ for some positive integer m. Then $z^m = ab^{-1}$ for some $a, b \in T$. Hence $z^m b \in T \cap P$. As $T \cap P = \emptyset$, we obtain a contradiction. It thus follows that $\mathrm{h}(\mathrm{gr}(z, H)) > \mathrm{h}(H) \geq \mathrm{h}(G) - 1$. Therefore $\mathrm{h}(\mathrm{gr}(z, H)) = \mathrm{h}(G)$ and, because G is polycyclic-by-finite, we get $[G : \mathrm{gr}(z, H)] < \infty$. Since $zs \in szF$ for every $s \in S \cap H$, it follows that $z(S \cap H) \subseteq (S \cap H)zF = (S \cap HF)z = (S \cap H)z$. As $H = (S \cap H)(S \cap H)^{-1}$, it follows that $zH = Hz$. Therefore $\Delta^+(H)$ has finitely many conjugates in G. Since $\Delta^+(H)$ is finite, we get that $\Delta^+(H) \subseteq \Delta^+(G) = \{1\}$ and thus $K[H]$ is a prime algebra. Hence, the localization $M_n(K[H])$ of $K[S]/K[Q]$ is prime. So $K[Q]$ is a prime ideal in $K[S]$. As $K[S]$ is prime and P is of height one, we get that $P = K[Q]$, as desired.

To prove the converse, let Q be a minimal prime ideal of S. Let P be an ideal of $K[S]$ maximal with respect to $S \cap P = Q$. Clearly P is a prime ideal of $K[S]$. Again by the remark before the theorem, we know that there exists an element z in S that belongs to Q and that is central modulo F. Then $zS = Sz$, so z is a normal element of $K[S]$. Since, by assumption $K[S]$ is a prime Noetherian algebra, the principal ideal theorem therefore yields a prime ideal P' of $K[S]$ that is of height one so that $z \in P'$ and $P' \subseteq P$. By the first part of the result,

$P' = K[S \cap P']$. Since $S \cap P'$ is a prime ideal of S contained in the minimal prime ideal $S \cap P = Q$, we get that $S \cap P' = Q$. So $K[Q] = P'$ is a height one prime ideal of $K[S]$. ∎

Recall that the rank $\mathrm{rk}(S)$ of a monoid S (not necessarily cancellative) is the supremum of the ranks of the free abelian subsemigroups of S. The dimension $\dim(S)$ of S is defined in the following way. By definition $\dim S = 0$ if $S = \{e\}$. If S has no zero element, then $\dim(S)$ is the maximal length n of a chain $Q_0 \subset Q_1 \subset Q_2 \subset \cdots \subset Q_n$, where $Q_0 = \emptyset$ and Q_i are prime ideals of S for $i > 0$ (note that primes are by definition different from S), or ∞ if such n does not exist. If $\{e\} \neq S$ has a zero element, then $\dim(S)$ is the maximal length n of such a chain with all Q_i $(i \geq 0)$ prime ideals of S, or ∞ if such n does not exist. The spectrum $\mathrm{Spec}(S)$ is the set of all prime ideals of S.

We give an easy example that shows that Theorem 1.1 can not be extended to semigroup algebras that are not Noetherian. Let $S = \{x^i y^j \mid i > 0, \ j \in \mathbb{Z}\} \cup \{1\}$, a submonoid of the free abelian group $\mathrm{gr}(x, y)$ of rank two. It is easy to see that for a given $i > 0$ and $j \in \mathbb{Z}$ the ideal $x^i y^j S$ contains the set $\{x^k y^j \mid k > i, \ j \in \mathbb{Z}\}$. Therefore $P = S \setminus \{1\}$ is nil modulo the ideal $x^i y^j S$. It follows that $S \setminus \{1\}$ is the only prime ideal of S. Moreover $\mathrm{rk}(S/P) = 0 < \mathrm{rk}(S) - 1$, while P is a minimal prime ideal of S. Clearly $K[P]$ is a prime ideal of $K[S]$. Let P' be the K-linear span of the set consisting of all elements of the form $x^i(y^j - y^k)$ with $i > 0$ and $j, k \in \mathbb{Z}$. Then P' is an ideal of $K[S]$ and $K[S]/P'$ is isomorphic to the polynomial algebra $K[x]$. Therefore $K[P]$ is a prime ideal of height two (note that $\mathrm{clKdim}(K[S]) = \mathrm{rk}(S) = 2$).

The latter equalities also follow from the following result, which will be needed later in the paper. Let T be a cancellative semigroup and K a field. If $K[T]$ satisfies a polynomial identity then $\mathrm{clKdim}(K[T]) = \mathrm{rk}(T) = \mathrm{GK}(K[T])$, [28, Theorem 23.4]. (By $\mathrm{GK}(R)$ we denote the Gelfand-Kirillov dimension of a K-algebra R.)

On the other hand, for Noetherian semigroup algebras one can also give an example (see Example 3.3) showing that Theorem 1.1 cannot be extended to prime ideals of height exceeding one.

In order to give some applications to the behaviour of prime ideals of S and those of $S \cap H$ (with H a normal subgroup of finite index in SS^{-1}) we prove the following technical lemma.

Lemma 1.2 *Let S be a submonoid of a polycyclic-by-finite group, say with a group of quotients G. Assume that S satisfies the ascending chain condition on right ideals. Then G has a poly-(infinite cyclic) normal subgroup H of finite index so that $S \cap H$ is G-invariant and $H/\mathrm{U}(S \cap H)$ is abelian. If G is abelian-by-finite then H can be chosen to be an abelian subgroup.*

Proof. It is well known that G contains a characteristic subgroup C that is poly-(infinite cyclic). Since S satisfies the ascending chain condition on right ideals, G contains a normal subgroup F so that G/F is abelian-by-finite and $F \subseteq S$. Hence $C \cap F$ is a normal subgroup of G that is poly-(infinite cyclic),

$C \cap F \subseteq \mathrm{U}(S)$ and $G/(C \cap F)$ is abelian-by-finite. It is thus sufficient to prove the result for the monoid $S/\sim_{C \cap F}$ and its group of quotients $G/(C \cap F)$. In other words we may assume that G is abelian-by-finite.

So, let A be a torsion free abelian and normal subgroup of finite index in G. Then $S \cap A$ is a finitely generated abelian monoid and its group of quotients is of finite index in G. Since S satisfies the ascending chain condition on right ideals, we also know [22] that for every $g \in S$ and $s \in S$ there exists a positive integer k so that $gs^k g^{-1} \in S$. Because $G = SZ(S)^{-1}$ (again by the remark before Theorem 1.1), the latter property holds for all $g \in G$. As $S \cap A$ is finitely generated abelian and G/A is finite, it follows that there exists a positive integer n so that

$$g(S \cap A)^{(n)} g^{-1} \subseteq S \cap A,$$

for all $g \in G$, where by definition $(S \cap A)^{(n)} = \{s^n \mid s \in S \cap A\}$. Hence

$$T = \bigcap_{g \in G} g(S \cap A)^{(n)} g^{-1}$$

is a G-invariant submonoid of S. Because A is the group of quotients of $S \cap A$, we get that each $g(S \cap A)^{(n)} g^{-1}$ has a group of quotients that is of finite index in G. Since there are only finitely many such conjugates, it is clear that TT^{-1} is of finite index in G. Hence, the result follows. ∎

The notion of height $\mathrm{ht}(P)$ of a prime ideal of a ring R is well known. We now define the height $\mathrm{ht}(Q)$ of a prime ideal Q of a monoid S. If S does not have a zero element then $\mathrm{ht}(Q)$ is the maximal length n of a chain $Q_0 \subset Q_1 \subset Q_2 \subset \cdots \subset Q_n = Q$, where $Q_0 = \emptyset$ and $Q_1, \ldots, Q_{n-1}$ are prime ideals of S. On the other hand, if S has a zero element, then $\mathrm{ht}(Q)$ is the maximal length of such a chain with all Q_i prime ideals of S, $i \geq 0$. If such n does not exist then we say that the height of P is infinite.

Corollary 1.3 *Let S be a submonoid of a polycyclic-by-finite group, say with a group of quotients G, and let K be a field. Assume that S satisfies the ascending chain condition on right ideals. If H is a torsion free normal subgroup of finite index in G, then*

1. *If P is a prime ideal of S then $P \cap H = Q_1 \cap \cdots \cap Q_n$, where $Q_1, \ldots, Q_n$ are all the primes of $S \cap H$ that are minimal over $P \cap H$. Furthermore, $P = M \cap S$ for any prime ideal M of $K[S]$ that is minimal over $K[P]$, $\mathrm{ht}(M) = \mathrm{ht}(K[Q_1]) = \cdots = \mathrm{ht}(K[Q_n])$ and $P = \mathcal{B}(S(Q_1 \cap \cdots \cap Q_n)S)$ (the prime radical of $S(Q_1 \cap \cdots \cap Q_n)S$). If, furthermore, $H \cap S$ is G-invariant then $Q_i = Q_1^g$ for some $g \in G$, $1 \leq i \leq n$.*

2. *If $S \cap H$ is G-invariant and $Q = Q_1$ is a prime ideal of $S \cap H$ then there exists a prime ideal P of S so that $P \cap H = Q_1 \cap Q_2 \cap \cdots \cap Q_m$, where $Q_1, \ldots, Q_m$ are all the prime ideals of $S \cap H$ that are minimal over $P \cap H$. One says that P lies over Q. Moreover, each $Q_i = Q^{g_i}$ for some $g_i \in G$.*

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3. **Incomparability** *Suppose $S \cap H$ is G-invariant, Q_1 and Q_2 are prime ideals of $S \cap H$, and P_1 and P_2 are prime ideals of S. If P_1 lies over Q_1 and P_2 lies over Q_2 so that $Q_1 \subseteq Q_2$ and $P_1 \subseteq P_2$, then $P_1 = P_2$ if and only if $Q_1 = Q_2$.*

4. **Going up** *Assume $S \cap H$ is G-invariant. Suppose Q_2 is a prime ideal of $S \cap H$ and P_2 is a prime ideal of S lying over Q_2.*

   (a) *If Q_1 is a prime ideal of $S \cap H$ containing Q_2 then there exists a prime ideal P_1 lying over Q_1 so that $P_2 \subseteq P_1$.*

   (b) *If P_1 is a prime ideal of S containing P_2 then there exists a prime ideal Q_1 of $S \cap H$ containing Q_2 so that P_1 lies over Q_1.*

5. **Going down** *Assume $S \cap H$ is G-invariant. Suppose Q_1 is a prime ideal of $S \cap H$ and P_1 is a prime ideal of S lying over Q_1.*

   (a) *If Q_2 is a prime ideal of $S \cap H$ contained in Q_1 then there exists a prime ideal P_2 lying over Q_2 so that $P_2 \subseteq P_1$.*

   (b) *If P_2 is a prime ideal of S contained in P_1 then there exists a prime ideal Q_2 of $S \cap H$ contained in Q_1 so that P_2 lies over Q_2.*

Proof. The algebra $K[S]$ has a natural gradation by the finite group G/H. Its homogeneous component of degree e (the identity of the group G) is the semigroup algebra $K[S \cap H]$. Let P be a prime ideal of S. Let M be a prime ideal of $K[S]$ minimal over $K[P]$. Note that then $K[S]/K[P]$ inherits a natural G/H-gradation, with component of degree e the algebra $K[S \cap H]/K[P \cap H]$. Because of Theorem 17.9 in [32] on going-up and down on prime ideals of rings graded by finite groups, one gets that $K[S \cap H] \cap M = P_1 \cap \cdots \cap P_n$, where $P_1, \ldots, P_n$ are all the prime ideals of $K[S \cap H]$ that are minimal over $K[P] \cap K[S \cap H] = K[P \cap H]$. Furthermore, $\mathrm{ht}(P_i) = \mathrm{ht}(M)$ for $1 \leq i \leq n$. Since H is torsion free, we know that $K[P_i \cap H]$ is a prime ideal of $K[S \cap H]$ (see the introduction). Since it clearly contains $K[P \cap H]$, it follows that $P_i = K[Q_i]$, with $Q_i = P_i \cap H$. Furthermore, since $K[Q]$ is a prime ideal in $K[S \cap H]$ for every prime ideal Q of $S \cap H$, it follows that $Q_1, \ldots, Q_n$ are all the prime ideals of $S \cap H$ minimal over $P \cap H$. So $K[S \cap H] \cap M = K[Q_1 \cap \cdots \cap Q_n]$. Because $K[S \cap H]/K[P \cap H]$ is Noetherian, we know that its prime radical is nilpotent. Hence $(Q_1 \cap \cdots \cap Q_n)^k \subseteq P \cap H$ for some positive integer k. Since H is of finite index in G, it then also follows that $M \cap S$ is an ideal of S that is nil modulo $S(P \cap H)S$. Since $K[S]$ is Noetherian, this yields that $(M \cap S)^l \subseteq S(P \cap H)S \subseteq P$, for some positive integer l. As P is a prime ideal, we therefore obtain that $M \cap S = P, P \cap H = Q_1 \cap \cdots \cap Q_n$ and $P = \mathcal{B}(S(Q_1 \cap \cdots \cap Q_n)S)$.

Assume now that, furthermore, $S \cap H$ is G-invariant. For an ideal I of $S \cap H$ put $I^{inv} = \bigcap_{g \in G} I^g$, the largest invariant ideal of $S \cap H$ contained in I. Clearly $SI^{inv} = I^{inv}S$ is an ideal of S and $SI^{inv} \cap (S \cap H) = I^{inv}$. It follows that

$$SQ_1^{inv} \cdots SQ_n^{inv} \subseteq P.$$

Hence $Q_i^{inv} \subseteq P \cap (S \cap H) = Q_1 \cap \cdots \cap Q_n$ for some i. Because $S \cap H$ is invariant, it follows that every Q_i^g is a prime ideal of $S \cap H$ (of the same height as Q_i). Hence, for every $1 \leq j \leq n$ there exists $g \in G$ with $Q_i^g = Q_j$. This proves the first part of the result.

To prove the second part, let Q be a prime ideal of $S \cap H$ and suppose that $S \cap H$ is G-invariant. Then, each Q^g is a prime ideal of $S \cap H$ and $\operatorname{ht}(Q) = \operatorname{ht}(Q^g)$. Now, if Q' is a prime ideal of $S \cap H$ containing Q^{inv}, then $Q^g \subseteq Q'$ for some $g \in G$. Hence, if Q' is a prime minimal over Q^{inv} then $Q^g = Q'$. Clearly, for every $h \in G$, the prime Q^h contains a prime ideal Q'' of $S \cap H$ minimal over Q^{inv}. Hence, by the previous, $Q'' = Q^g \subseteq Q^h$, for some g. Since $\operatorname{ht}(Q^g) = \operatorname{ht}(Q^h)$, it thus follows that $Q^h = Q''$. So, we have shown that the ideals Q^g are precisely the prime ideals of $S \cap H$ that are minimal over Q^{inv}. Since H is torsion free we thus get (again see the introduction) that the ideals $K[Q^g]$ are precisely the prime ideals of $K[S \cap H]$ that are minimal over $K[Q^{inv}]$. Since $SQ^{inv} = Q^{inv}S$ is an ideal of S, the algebra $K[S]/K[SQ^{inv}]$ has a natural G/H-gradation, with $K[S \cap H]/K[Q^{inv}]$ as component of degree e. Hence, by [32, Theorem 17.9], there exists a prime ideal M of $K[S]$ that is minimal over $K[SQ^{inv}]$ and $M \cap K[S \cap H] = K[Q] \cap P_2 \cap \cdots \cap P_m$, where $K[Q], P_2, \ldots, P_m$ are all the prime ideals that are minimal over $M \cap K[S \cap H]$ (and these ideals are of the same height as M). Hence each P_i is minimal over $K[Q^{inv}]$ and thus is of the form $K[Q^g]$ for some g. It follows that $P = M \cap S$ is a prime ideal of S so that $P \cap H$ is an intersection of Q and some of the prime ideals Q^g, with $g \in G$. This proves part two.

Parts (3), (4) and (5) are now immediate consequences of parts (1) and (2) and of the corresponding results on going up and down for rings graded by finite groups (see for example [32, Theorem 17.9]). ∎

A fundamental result (see [32, Theorem 19.6]) on the group algebra $K[G]$ of a polycyclic-by-finite group says that $\operatorname{clKdim}(K[G]) = \operatorname{pl}(G)$. For the definition on the plinth length $\operatorname{pl}(G)$ of G we refer the reader also to [32]. In the following result we determine relations between the considered invariants of S and $K[S]$ for Noetherian semigroup algebras $K[S]$ of submonoids S of polycyclic-by-finite groups.

Corollary 1.4 *Let S be a submonoid of a polycyclic-by-finite group and let K be a field. Assume that S satisfies the ascending chain condition on right ideals and let G be its group of quotients. Let F be a normal subgroup of G so that $F \subseteq S$, $\operatorname{U}(S)/F$ is finite and G/F is abelian-by-finite. The following properties hold.*

1. $\dim(S) = \operatorname{rk}(G/F)$.

2. $\dim(S) = \dim(S \cap W)$ *for any normal subgroup W of G of finite index.*

3. $\operatorname{Spec}(S)$ *is finite.*

4. $\operatorname{clKdim}(K[S]) = \dim(S) + \operatorname{pl}(\operatorname{U}(S))$.

Proof. Because of Lemma 1.2, the group G has a poly-(infinite cyclic) normal subgroup H of finite index so that $S \cap H$ is G-invariant and H/F is abelian for some normal subgroup F of H that is contained in S. Parts (1-5) of Corollary 1.3 easily yield that $\dim(S) = \dim(S \cap H)$. It also easily is verified that $\dim(S \cap H) = \dim(T)$ where $T = (S \cap H)/ \sim_F$. Clearly TT^{-1} is abelian and $TT^{-1} = H/F$ because $[G : H] < \infty$. Since $U(S)/F$ is finite, we obtain that $U(T)$ is finite. Hence $\dim(T) = \mathrm{rk}(T)$ (see [22]). So part (1) follows.

To prove part (2) let W be a normal subgroup of finite index in G. Then, $S \cap W$ inherits the assumptions on S and W is the group of quotients of $S \cap W$. So, from part (1) we obtain that $\dim(S) = \mathrm{rk}(G/F) = \mathrm{rk}(W/(F \cap W)) = \dim(S \cap W)$ and thus $\dim(S) = \dim(S \cap W)$.

It easily is seen that there is a natural bijective map between $\mathrm{Spec}(S)$ and $\mathrm{Spec}(S/ \sim_F)$. Hence to prove part (3) we may assume that SS^{-1} is abelian-by-finite. Because of Lemma 1.2 and part (1) of Corollary 1.3, we obtain that $\mathrm{Spec}(S)$ is finite if the corresponding property holds for finitely generated abelian monoids $A = \langle a_1, \ldots, a_n \rangle$. This is obviously satisfied. Indeed, if $P \in \mathrm{Spec}(A)$ then $A \setminus P$ is a submonoid of A and $a_i \in P$ for some i. It follows that $A \setminus P$ is generated by a proper subset of $\{a_1, \ldots, a_n\}$, whence $|\mathrm{Spec}(A)| < \infty$.

Finally, we prove part (4). From [22, Corollary 4.4] we know that

$$\mathrm{clKdim}(K[S]) = \mathrm{rk}(G/F) + \mathrm{pl}(U(S)).$$

So the statement follows at once from part (1). ∎

In the following proposition it is shown that the prime spectra of S and $S/ \sim_{\Delta^+(SS^{-1})}$ can be identified.

Proposition 1.5 *Let S be a submonoid of a polycyclic-by-finite group $G = SS^{-1}$ and assume that S satisfies the ascending chain condition on right ideals. Let $H = \Delta^+(G)$, the maximal finite normal subgroup of G. Let $\sim$ be the congruence relation on S determined by H, that is, $s \sim t$ if and only if $sH = tH$. Then the map $\mathrm{Spec}(S) \longrightarrow \mathrm{Spec}(S/\sim)$, defined by $P \mapsto P/\sim$, is a bijection.*

Proof. Let $\varphi : S \longrightarrow S/\sim \subseteq G/H$ be the natural epimorphism. Let P be a prime ideal of S.

We claim that $P = \varphi^{-1}(\varphi(P))$. One inclusion is obvious. To prove the converse, let $x \in \varphi^{-1}(\varphi(P))$. Then there exists $p \in P$ such that $x \sim p$. So $x = ph$ for some $h \in H$. Let $s \in S$. Then

$$(hsp)^n = hh^{sp}h^{(sp)^2} \cdots h^{(sp)^{n-1}}(sp)^n,$$

for every $n \geq 1$. Because $h \in H$, there exists a positive integer k so that $h^{(sp)^k} = h$. Define $n = (k-1)|H| + 1$. Then

$$(hsp)^n = (hh^{sp}h^{(sp)^2} \cdots h^{(sp)^{k-1}})^{|H|}(sp)^n = (sp)^n \in P.$$

It follows that xS is nil modulo P. Because $K[S]$ is Noetherian, from [17, Theorem 5.18] we get that $x \in P$. This proves the claim.

The claim easily implies that $\varphi(P)$ is a prime ideal of S and the statement follows. ∎

We can now prove for the semigroups under consideration an analogue of Schelter's theorem on prime affine algebras that satisfy a polynomial identity; this on its turn yields the catenary property (see for example [27, Theorem 13.10.12 and Corollary 13.10.13]).

Proposition 1.6 *Let S be a submonoid of a polycyclic-by-finite group, say with a group of quotients G. Assume that S satisfies the ascending chain condition on right ideals. Let P be a prime ideal of S. Then $\dim(S/P) + \mathrm{ht}(P) = \dim(S)$. Furthermore, if $\mathrm{U}(S)$ is finite, then $\dim(S/P) = \mathrm{rk}(S/P)$.*

Proof. Again let F be a normal subgroup of G so that $F \subseteq \mathrm{U}(S)$, $[\mathrm{U}(S) : F] < \infty$, $S \cap F$ is finitely generated and G/F is abelian-by-finite. Because of the natural bijection between $\mathrm{Spec}(S)$ and $\mathrm{Spec}(S/ \sim_F)$, we may replace S by $S/ \sim_F$ and thus we may assume that G is abelian-by-finite and $\mathrm{U}(S)$ is finite. Hence, by Lemma 1.2, G contains a normal torsion free abelian subgroup A so that $[G : A] < \infty$, $T = S \cap A$ is finitely generated and G-invariant.

We now first prove that $\dim(T/Q) + \mathrm{ht}(Q) = \dim(T)$ for a prime ideal Q in the abelian monoid T. Since A is torsion free, we know (see the introduction) that $K[Q]$ is a prime ideal of $K[T]$ (which is of height one if Q is a minimal prime of T). Hence (by Schelter's result for finitely generated commutative algebras) $\mathrm{clKdim}(K[T]) = \mathrm{clKdim}(K[T]/K[Q]) + \mathrm{ht}(K[Q])$. Note that $T \setminus Q$ is a submonoid of A and $\mathrm{clKdim}(K[T]/K[Q]) = \mathrm{clKdim}(K[T \setminus Q])$ (we will several times use this fact without specific reference). Consequently, by Corollary 1.4, $\dim(T) = \dim(T \setminus Q) + \mathrm{ht}(K[Q])$. Since $\dim(T \setminus Q) = \dim(T/Q)$, we need to prove that $\mathrm{ht}(K[Q]) = \mathrm{ht}(Q)$. We prove this by induction on $\mathrm{ht}(Q)$ (note that by Corollary 1.4 the prime spectrum of T is finite and hence every prime contains a minimal prime ideal of T). If $\mathrm{ht}(Q) = 1$ then the statement holds. So, assume $\mathrm{ht}(Q) > 1$. Let Q_1 be a prime of height one contained in Q. Then, by the induction hypothesis, $\mathrm{ht}(Q/Q_1) = \mathrm{ht}(K[Q/Q_1])$. Since $K[(T/Q_1)/(Q/Q_1)] \cong K[T/Q]$, we thus get from Schelter's result that $\mathrm{clKdim}(K[T]) - 1 = \mathrm{clKdim}(K[T/Q_1]) = \mathrm{clKdim}(K[T/Q]) + \mathrm{ht}(Q/Q_1) = \mathrm{clKdim}(K[T]) - \mathrm{ht}(K[Q]) + \mathrm{ht}(Q/Q_1)$. Hence $\mathrm{ht}(K[Q]) = \mathrm{ht}(Q/Q_1) + 1 \leq \mathrm{ht}(Q)$. Since $K[M]$ is prime in $K[T]$ if M is prime in T, it is clear that $\mathrm{ht}(Q) \leq \mathrm{ht}(K[Q])$. Hence we obtain that $\mathrm{ht}(K[Q]) = \mathrm{ht}(Q)$, as desired.

Now let P be a prime ideal of S. Because of Corollary 1.3, $\dim(S) = \dim(S \cap A)$, $\dim(S/P) = \dim(T/Q)$ and $\mathrm{ht}(P) = \mathrm{ht}(Q)$, where Q is a prime ideal of T and P lies over Q. From the previous it thus follows that

$$\dim(S/P) + \mathrm{ht}(P) = \dim(S).$$

So, only the last part of the statement of the result remains to be proven. Let $Q_1 = Q$ and write $P \cap T = Q_1 \cap \cdots \cap Q_n$, where $Q_1, \ldots, Q_n$ are all primes minimal

over $P \cap T$. We know that $\mathrm{clKdim}(K[T/Q_i]) = \mathrm{rk}(T/Q_i)$. Furthermore, because TT^{-1} is torsion free and $K[T]$ is Noetherian, we also know that $K[Q_i]$ is a prime ideal of $K[T]$ that is minimal over $K[P \cap T]$ (see the introduction). Hence, $K[Q_i/(P \cap T)]$ is a minimal prime ideal in the finitely generated commutative algebra $K[T/(P \cap T)]$. It follows that

$$
\begin{aligned}
\mathrm{clKdim}(K[T/(P \cap T)]) &= \mathrm{clKdim}(K[T/Q_i]) + \mathrm{ht}(K[Q_i/(P \cap T)]) \\
&= \mathrm{clKdim}(K[T/Q_i]) = \mathrm{rk}(T/Q_i). \qquad (1)
\end{aligned}
$$

Since $K[T/(P \cap T)]$ is a finitely generated commutative algebra, it is well known that $\mathrm{clKdim}(K[T/(P \cap T)]) = \mathrm{GK}(K[T/(P \cap T)])$ (see for example [26, Theorem 4.5]). From [28, Theorem 23.14]) it follows that

$$
\mathrm{rk}(T/(P \cap T)) = \mathrm{clKdim}(K[T/(P \cap T)]).
$$

Using (1) we thus get

$$
\begin{aligned}
\mathrm{rk}(T/(P \cap T)) &= \mathrm{clKdim}(K[T]/(K[Q_1] \cap \cdots \cap K[Q_n])) \\
&= \sup\{\mathrm{clKdim}(K[T/Q_i]) \mid 1 \le i \le n\} \\
&= \mathrm{clKdim}(K[T/Q]) = \mathrm{rk}(T/Q).
\end{aligned}
$$

Since TT^{-1} is of finite index in SS^{-1}, it follows that

$$
\mathrm{rk}(S/P) = \mathrm{rk}(T/(T \cap P)) = \mathrm{rk}(T/Q).
$$

Because $\mathrm{U}(T/Q)$ is finite, Corollary 1.4 yields that $\mathrm{rk}(T/Q) = \dim(T/Q)$ and thus we get that

$$
\begin{aligned}
\mathrm{rk}(S/P) = \mathrm{rk}(T/Q) &= \dim(T/Q) = \dim(T) - \mathrm{ht}(Q) \\
&= \dim(S) - \mathrm{ht}(P) = \dim(S/P).
\end{aligned}
$$

This finishes the proof. ∎

2 Maximal Orders

In this section we describe when a semigroup algebra $K[S]$ of a cancellative submonoid S of a polycyclic-by-finite group G is a prime Noetherian maximal order that satisfies a polynomial identity. In case G is torsion free such a result was obtained in [21] and in case $S = G$ this was done by Brown in [3, 4] (even without the restriction that $K[S]$ has to be PI).

For completeness' sake we recall some notation and terminology on (maximal) orders. We state these in the semigroup context (see for example [16] and [34]) as these are basically the same as in the more familiar ring case. A cancellative monoid S which has a left and right group of quotients G is called an *order*. Such a monoid S is called a *maximal order* if there does not exist

a submonoid S' of G properly containing S and such that $aS'b \subseteq S$ for some $a, b \in G$. For subsets A, B of G we define $(A :_l B) = \{g \in G \mid gB \subseteq A\}$ and by $(A :_r B) = \{g \in G \mid Bg \subseteq A\}$. Note that S is a maximal order if and only if $(I :_l I) = (I :_r I) = S$ for every *fractional ideal* I of S. The latter means that $SIS \subseteq I$ and $cI, Id \subseteq S$ for some $c, d \in S$. If S is a maximal order, then $(S :_l I) = (S :_r I)$ for any fractional ideal I; we simply denote this fractional ideal by $(S : I)$ or by I^{-1}. Recall that then I is said to be *divisorial* if $I = I^*$, where $I^* = (S : (S : I))$. The divisorial product $I * J$ of two divisorial ideals I and J is defined as $(IJ)^*$. Also recall that a fractional ideal is said to be *invertible* if $IJ = JI = S$ for some fractional ideal J of S. In this case $J = I^{-1}$ and I is a divisorial ideal.

Recall then that (see for example [34]) a cancellative monoid S is said to be a Krull order if and only if S is a maximal order satisfying the ascending chain condition on divisorial integral ideals (the latter are the fractional ideals contained in S). In this case the set $D(S)$ of divisorial fractional ideals is a free abelian group for the $*$ operation. If SS^{-1} is abelian-by-finite then (as said before, see [25, Lemma 1.1]) every ideal of S contains a central element and it follows that the minimal primes of S form a free basis for $D(S)$, [34].

Similarly a prime Goldie ring R is said to be a Krull order if R is a maximal order that satisfies the ascending chain condition on divisorial integral ideals. Although there are several notions of noncommutative Krull orders, for rings satisfying a polynomial identity all these notions are the same.

In the next theorem we collect some of the essential properties of these orders. For details we refer the reader to [5, 6]. For a ring R and an Ore set C of regular elements in R we denote by R_C the classical localization of R with respect to C. The classical ring of quotients of a prime Goldie ring R is denoted by $Q_{cl}(R)$. The prime spectrum of R is denoted by $\mathrm{Spec}(R)$, the set of height one prime ideals of R by $X^1(R)$.

Theorem 2.1 *Let R be a prime Krull order satisfying a polynomial identity. Then the following properties hold.*

1. *The divisorial ideals form a free abelian group with basis $X^1(R)$, the height one primes of R.*

2. *If $P \in X^1(R)$ then $P \cap Z(R) \in X^1(Z(R))$, and furthermore, for any ideal I of R, $I \subseteq P$ if and only if $I \cap Z(R) \subseteq P \cap Z(R)$.*

3. *$R = \bigcap R_{Z(R) \backslash P}$, where the intersection is taken over all height one primes of R, and every regular element $r \in R$ is invertible in almost all (that is, except possibly finitely many) localizations $R_{Z(R) \backslash P}$. Furthermore, each $R_{Z(R) \backslash P}$ is a left and right principal ideal ring with a unique nonzero prime ideal.*

4. *For a multiplicatively closed set of ideals $\mathcal{M}$ of R, the (localized) ring $R_{\mathcal{M}} = \{q \in Q_{cl}(R) \mid Iq \subseteq R, \text{ for some } I \in \mathcal{M}\}$ is a Krull order, and*

$$R_{\mathcal{M}} = \bigcap R_{Z(R) \backslash P},$$

13

where the intersection is taken over those height one primes P for which
$R_{\mathcal{M}} \subseteq R_{Z(R)\setminus P}$.

Next we prove some necessary condition for $K[S]$ to be a prime Noetherian maximal order that satisfies a polynomial identity.

Lemma 2.2 *Let S be a submonoid of an abelian-by-finite group $G = SS^{-1}$ and let K be a field. Let A be an abelian subgroup that is normal and of finite index in G and let P be a minimal prime ideal of S. The following properties hold.*

1. *If S is a maximal order then $S \cap A$ is G-invariant and $S \cap A$ is a maximal order in its group of quotients.*

2. *If $K[S]$ is a prime Noetherian maximal order, then S is a maximal order, $S \cap A$ and $P \cap A$ are G-invariant.*

Proof. By [21, Lemma 2.1], if S is a maximal order then $S \cap A$ is G-invariant and it is a maximal order in its group of quotients. Assume now that $K[S]$ is a prime Noetherian maximal order. It is straightforward (as in the proof of Lemma 3.3 in [21]) to verify that then S is a maximal order, and thus $S \cap A$ is G-invariant. Let P be a minimal prime ideal of S. Of course, $A \subseteq \Delta(G)$ and thus $P \cap A = (P \cap \Delta(G)) \cap (A \cap S)$. Hence to prove that $P \cap A$ is G-invariant, we may assume in the remainder that $A = \Delta(G)$. Indeed, since $K[S]$ is prime, by [28, Theorem 7.19], $K[G]$ is prime. Then, from [31, Theorem 4.2.10] it follows that $\Delta(G)$ is abelian. Now, by Corollary 1.3, $P \cap A = Q_1 \cap \cdots \cap Q_n$, an intersection of minimal primes of A that are G-conjugate. We need to show that $\{Q_1, \ldots, Q_n\} = \{Q_1^g \mid g \in G\}$. Suppose the contrary, so assume Q' is a minimal prime of $S \cap A$ that is conjugate to Q_1 but is different from all Q_i, for $1 \leq i \leq n$. Then by Corollary 1.3, there exists a prime ideal P' of S so that $P' \cap A = Q' \cap Q'_2 \cap \cdots \cap Q'_m$ for some minimal primes $Q'_2, \ldots, Q'_m$ of $S \cap A$. Because of Theorem 1.1, both $K[P]$ and $K[P']$ are distinct height one prime ideals of $K[S]$. As, by assumption, $K[S]$ is a prime PI Noetherian maximal order, it follows from Theorem 2.1 that there exists a central element of $K[S]$ that belongs to $K[P']$ but not to $K[P]$. Since central elements of $K[S]$ are linear combinations of finite conjugacy class sums of $K[G]$ we get that P' contains a G-conjugacy class C that does not belong to P. Since $A = \Delta(G)$, we thus get that $C \subseteq A$. Hence, $C \subseteq Q'$. As Q' is a G-conjugate of Q_1 it follows that $C \subseteq Q_1 \cap \cdots \cap Q_n = P \cap A$, a contradiction. ∎

We need one more lemma in order to prove the main theorem of this section. If $\alpha = \sum_{s \in S} k_s s$ (with each $k_s \in K$) then we denote by $\operatorname{supp}(\alpha) = \{s \in S \mid k_s \neq 0\}$ the support of α.

Lemma 2.3 *Let S be a submonoid of an abelian-by-finite group $G = SS^{-1}$. Let K be a field and suppose $K[S]$ is Noetherian. Let A be a normal abelian subgroup of finite index in G. Assume that P is a prime ideal of S so that $K[P]$ is a prime ideal of $K[S]$ and $P \cap A$ is G-invariant. Also assume that $S \cap A$ is G-invariant. If J is an ideal of S not contained in P then $J \cap A$ contains a G-conjugacy class D such that $D \not\subseteq P$ (and thus clearly $D \subseteq (J \cap A) \setminus P$).*

Proof. We may assume that J is a proper ideal of S. The prime algebra $R = K[S]/K[P]$ has a natural G/A-gradation, with identity component $R_e = K[S \cap A]/K[P \cap A]$, a semiprime commutative algebra. Let L be a nonzero ideal of R. It then follows from [10, Theorem 1.7] that $r_e \in L \cap R_e$, for some regular element r_e of R_e. Because of the assumptions, G acts by conjugation in R_e. Clearly, r has only finitely many such conjugates, say $r_1, \ldots, r_m$ and $r_1 \cdots r_m \in L \cap R_e$ is central and nonzero. So, L contains a non-trivial element in $R_e \cap \mathrm{Z}(R)$.

We apply the above to the ideal $L = (K[J] + K[P])/K[P]$. So, let $\alpha \in K[J \cap A]$ be such that the image $\overline{\alpha} \in K[S]/K[P]$ is nonzero and lies in the center. So α is regular modulo $K[P]$. We may assume that $\mathrm{supp}(\alpha) \cap P = \emptyset$. Note also that $1 \notin \mathrm{supp}(\alpha)$. Write $\alpha = \alpha_1 + \cdots + \alpha_q$, where α_i have supports contained in different G-conjugacy classes. Then $g\alpha_1 + \cdots + g\alpha_q - (\alpha_1 g + \cdots + \alpha_q g) \in K[P]$ for every $g \in S$. Clearly, $g\alpha_i$ and $\alpha_j g$ have disjoint supports if $i \neq j$. So we must have $g\alpha_i - \alpha_i g \in K[P]$ for every i. Then every α_i also lies in the center modulo $K[P]$ and $\mathrm{supp}(\alpha_i)$ is contained in a G-conjugacy class D_i. By the hypothesis, $D_i \subseteq (S \cap A) \setminus P$. Hence, replacing α by α_1, we may assume that $\alpha = \alpha_1$. Write $D = D_1$.

Because of the assumptions and Corollary 1.3, $P \cap A = Q_1 \cap \cdots \cap Q_n$ where $Q_1, \ldots, Q_n$ is a full orbit of conjugate primes in $S \cap A$. For $1 \leq i \leq n$ let $A_i = \left(\bigcap_{j,\, j \neq i} Q_j \right) \setminus Q_i$. Each A_i is a subsemigroup of $A \setminus P$. Each $g \in G$ permutes the sets $A_1, \ldots, A_n$ (by conjugation). Let $I = \bigcup_{i=1}^n A_i \cup (P \cap A)$. Then I is G-invariant, $SI = IS$ and $SI \cap A = I$. Replacing J by $J \cap SI$ we may assume that $J \subseteq SI$ and thus also $\alpha \in K[J \cap A] \subseteq K[SI \cap A] = K[I]$. Hence, we can write $\alpha = \gamma_1 + \cdots + \gamma_n$, with $\mathrm{supp}(\gamma_i) \subseteq A_i$ for $i = 1, \ldots, n$. Notice that $\mathrm{supp}(\gamma_i) \neq \emptyset$ for every i. Indeed, suppose the contrary, that is, assume $\gamma_i = 0$ for some i. Then, $\alpha A_i \subseteq P$, in contradiction with the regularity of α modulo $K[P]$. So, indeed, $\mathrm{supp}(\gamma_i) \neq \emptyset$ for every i. Let $E_i = D \cap A_i$. Since also $\mathrm{supp}(\gamma_i) \subseteq D$, we get that $\emptyset \neq \mathrm{supp}(\gamma_i) \subseteq E_i$. Put $a_i = \prod_{x \in E_i} x$. Clearly $a_i \in A_i$. It follows that $a_1 + \cdots + a_n \in K[J]$. Let $1 \leq i \leq n$ and $g \in G$. Then $g^{-1}A_i g = A_j$ for some j. Hence $g^{-1}E_i g = E_j$ and thus $g^{-1}a_i g = a_j$. So, conjugation by elements of G permutes $a_1, \ldots, a_n$. Therefore the result follows. ∎

Recall from [3, 32] that a group G is dihedral free if, for every subgroup D of G isomorphic to the infinite dihedral group, the normalizer $N_G(D)$ of D in G has infinite index.

Theorem 2.4 *Let K be a field and let S be a submonoid of a finitely generated abelian-by-finite group. Let A be an abelian normal subgroup of finite index in $G = SS^{-1}$. The following conditions are equivalent.*

1. *$K[S]$ is a prime Noetherian maximal order.*

2. *S is a maximal order that satisfies the ascending chain condition on right ideals, $\Delta^+(G) = \{1\}$, G is dihedral free and for every minimal prime ideal P of S the set $A \cap P$ is G-invariant.*

Proof. Assume that $K[S]$ is a prime Noetherian maximal order. Then, by Theorem 2.1, the localization $K[G] = K[SZ(S)^{-1}]$ of $K[S]$ also is a maximal order. Because of Brown's result on the description of group algebras of polycyclic-by-finite groups that are maximal orders, the latter holds if and only if $\Delta^+(G) = \{1\}$ and G is dihedral free. Lemma 2.2 then yields that the other conditions listed in (2) hold as well.

Conversely, assume that condition (2) holds. The assumption on $\Delta^+(G)$ yields that $K[G]$ and thus $K[S]$ is prime. Because S is a maximal order, Lemma 2.2 gives that $S \cap A$ is G-invariant for any abelian normal subgroup A of G of finite index. To prove that $K[S]$ is a maximal order one can follow the lines of the proof of Theorem 3.5 in [21]. We now give a simplified proof.

We begin by showing that if P is a minimal prime ideal of S then the localized ring $R = K[S]((Z(K[S]) \cap K[A]) \setminus K[P])^{-1}$ is a maximal order. To do so, we show that R is a local ring with unique maximal ideal RP and so that RP is invertible and every proper nonzero ideal of R is of the form $(RP)^n$ for some positive integer n. First we show that RP is the only height one prime ideal of R. Of course if Q' is a height one prime ideal of R then $Q = K[S] \cap Q'$ is a height one prime ideal of $K[S]$ that does not intersect $(Z(K[S]) \cap K[A]) \setminus K[P]$. Because of Theorem 1.1, either $Q = K[S \cap Q]$ or $S \cap Q = \emptyset$. Because of Lemma 2.3, the former implies that $S \cap Q \subseteq P$ and thus $Q = K[P]$, as desired. So assume $S \cap Q = \emptyset$, or equivalently, $A \cap Q = \emptyset$. Because $S \cap A$ is G-invariant and Q does not contain homogeneous elements, it follows (see for example Lemma 7.1.4 in [24]) that $\overline{Q} = \bigcap_{g \in G} g^{-1}(Q \cap K[A])g$ is not contained in $K[P \cap A]$. As $\overline{Q}$ is G-invariant, we get that $K[S]\overline{Q}$ is an ideal of $K[S]$. So $(K[S]\overline{Q} + K[P])/K[P]$ is a nonzero ideal of the Noetherian algebra $K[S]/K[P]$. This algebra has a natural G/A-gradation, with component of degree e the semiprime algebra $K[S \cap A]/K[P \cap A]$. By Theorem 1.7 in [10], the ring $K[S]/K[P]$ has a classical ring of quotients that is obtained by inverting the regular elements of $K[S \cap A]/K[P \cap A]$. Hence the ideal $(K[S]\overline{Q} + K[P])/K[P]$ contains a regular element $\overline{\alpha}$ that is contained in $K[S \cap A]/K[P \cap A]$. Since $K[S \cap A]$ and $K[P \cap A]$ are G-invariant, conjugation induces an action of G on $K[S \cap A]/K[P \cap A]$. Hence the product of the finitely many conjugates of $\overline{\alpha}$ also belongs to $K[S \cap A]/K[P \cap A]$. Since this element is central, we thus may assume that $\overline{\alpha}$ also is central in $K[S]/K[P]$ and clearly $\overline{\alpha} \in (\overline{Q} + K[P \cap A])/K[P \cap A]$. Write $\overline{\alpha} = \gamma + K[P \cap A]$ for some $\gamma \in \overline{Q}$. Let β be the product of the distinct conjugates of γ. Then $\beta + K[P \cap A] = \gamma^m + K[P \cap A]$ for some positive integer m. Since $K[P \cap A]$ is a semiprime ideal in $K[S \cap A]$, it follows that $\beta \notin K[P \cap A]$. Hence $\beta \in (Q \cap Z(K[S])) \setminus K[P]$, a contradiction. This implies that indeed RP is the only height one prime ideal of R.

Because of Lemma 2.2, $S \cap A$ is a finitely generated maximal order. Hence we know that $K[S \cap A]$ is a Noetherian maximal order ([1, 2]) and thus it is well known (or use Theorem 2.1) that $K[S \cap A]((Z(K[S] \cap K[A]) \setminus P)^{-1}$ is a Noetherian maximal order with only finitely many height one prime ideals. Hence it is a principal ideal domain (see for example [13]) and thus it has prime dimension one. As this is the component of degree e of the G/A-graded ring R, from [32, Theorem 17.9] it follows that R also has dimension one. Hence RP is

the only maximal ideal of R. As R also is a PI algebra, we then obtain that RP is the Jacobson radical of R and thus, by [17, Theorem 8.12], $\bigcap_n (RP)^n = \{0\}$. Also note that $P(S : P)$ is an ideal of S that is not contained in P. Hence $RP(S : P) = R$ and thus the unique maximal ideal RP is an invertible ideal of R. It then easily follows that every proper nonzero ideal of R is of the form $(RP)^n$ for some unique positive integer n. This proves the desired properties of R.

Let $\alpha = \sum_{i=1}^n k_i g_i \in K[G]$, where $0 \neq k_i \in K$ and $g_i \in G$ for each $1 \leq i \leq n$ and $g_i \neq g_j$ for $i \neq j$ We now show that if $\alpha \in K[G] \cap \bigcap_P K[S]((Z(K[S]) \cap K[A]) \setminus K[P])^{-1}$ (where the intersection runs over all minimal primes P of S) then $\alpha \in K[S]$. We prove this by induction on n. If $n = 1$ then $\alpha = k_1 g_1$. For each minimal prime P of S there then exists a central element δ of $K[S]$ that belongs to $K[A] \setminus P$ so that $\delta k_1 g_1 \in K[S]$. Hence there is a G-conjugacy class $C(P)$ so that $C \subseteq (S \cap A) \setminus P$ and $C(P)g_1 \subseteq S$. Let $C = \bigcup_P C(P)$. Then $SCg_1 \subseteq S$ and SC is an ideal of S that is not contained in any of the minimal prime ideals of S. Since S is a maximal order, it follows that $g_1 \in S$, as desired. Now assume $n > 1$. Since $\bigcap_P K[S]((Z(K[S]) \cap K[A]) \setminus K[P])^{-1}$ is a G/A-graded ring, the induction hypothesis yields that we may assume that α is G/A-homogeneous, that is, each $g_i g_j^{-1} \in A$. Since this statement holds for any normal abelian subgroup of G of finite index and because A is residually finite, we get that $g_i = g_j$ for all $i = j$. Hence we may assume $n = 1$ and thus by the above $\alpha \in K[S]$.

So we have shown that $K[S] = K[G] \cap \bigcap_P K[S]((Z(K[S]) \cap K[A]) \setminus K[P])^{-1}$. Recall that by Theorem F in [3], since G is dihedral-free, $K[G]$ is a maximal order. Since also each $K[S]((Z(K[S]) \cap K[A]) \setminus K[P])^{-1}$ is a maximal order and a central localization of $K[S]$, it follows that $K[S]$ is a maximal order. This finishes the proof. ∎

3   Constructing examples

Theorem 2.4 reduces the problem of determining when $K[S]$ is a prime Noetherian maximal order to the algebraic structure of S. It hence provides a strong tool for constructing new classes of such algebras. For some examples the required conditions on S can easily be verified, but on the other hand, for some examples this still requires substantial work. In this section this is illustrated with some concrete constructions.

A first class of examples consists of algebras defined by monoids of I-type (see [14, 24]). Recall that, in particular, these are quadratic algebras R with a presentation defined by n generators $x_1, \ldots, x_n$ and with $\binom{n}{2}$ relations of the form $x_i x_j = x_k x_l$ so that every word $x_i x_j$ appears at most once in one of the defining relations. Clearly $R = K[S]$, where S is the monoid defined by the same presentation. It turns out that S has a group of quotients G that is torsion free and has a free abelian subgroup A of finite index. Furthermore, for any minimal prime ideal P of S one has that $P = Ss = sS$, $P \cap A = (S \cap A)a$,

for some $s \in S$, $a \in A$, and $P \cap A$ is G-invariant. Using Theorem 2.4 we then immediately recover the known result that R is a maximal order. The only noncommutative algebra of such type which is generated by two elements is $K\langle x, y \mid x^2 = y^2 \rangle$ ([14]). A related example on three generators that is not of this type is $K\langle x, y, z \mid x^2 = y^2 = z^2, zx = yz, zy = xz \rangle$. As an application of Theorem 2.4 one can show by elementary calculations that this algebra also is a prime Noetherian PI maximal order.

In the remainder of this section we discuss in full detail one more construction that illustrates Theorem 2.4 but also shows that certain assumptions in Theorem 1.1 are essential. Before this, we establish a useful general method for constructing nonabelian submonoids of abelian-by-finite groups that are maximal orders, starting from abelian maximal orders.

Proposition 3.1 *Let A be an abelian normal subgroup of finite index in a group G. Suppose that B is a submonoid of A so that $A = BB^{-1}$ and B is a finitely generated maximal order. Let S be a submonoid of G such that $G = SS^{-1}$ and $S \cap A = B$. Then S is a maximal order that satisfies the ascending chain condition on right ideals if and only if S is maximal among all submonoids T of G with $T \cap A = B$.*

Proof. First suppose S is a maximal order that satisfies the ascending chain condition on right ideals. Suppose that $S \subseteq T \subseteq G$ for a submonoid T of G such that $S \cap A = B = T \cap A$. By assumption, B is finitely generated. Since also A is normal and of finite index in G, it thus follows (see the introduction) that $K[T]$ is Noetherian and it is a finitely generated right $K[B]$-module. Thus T satisfies the ascending chain condition on one-sided ideals and $T = \bigcup_{i=1}^{n} t_i B$ for some $n \geq 1$ and $t_i \in T$. Since G is finitely generated and abelian-by-finite, we know that for every i there exists $z_i \in \mathrm{Z}(S)$ such that $z_i t_i \in S$. Let $z = z_1 \cdots z_k$. Then $z t_i \in S$ and therefore $zT = \bigcup_i z t_i (S \cap A) \subseteq S$. Since S is a maximal order, this implies that $T = S$. So, we have shown that S is maximal among all submonoids $T \subseteq SS^{-1}$ such that $T \cap A = S \cap A$.

Conversely, assume that $S \subseteq G$ is maximal among all submonoids T of G with $T \cap A = B$. As above, because $S \cap A$ is finitely generated, S satisfies the ascending chain condition on right ideals. Suppose that $T \subseteq G$ is a submonoid such that $S \subseteq T$ and $gTh \subseteq S$ for some $g, h \in G$. There exist $s, t \in S$ such that $sg, ht \in \mathrm{Z}(G) \cap B$. So $sgTht \subseteq S$ and therefore $Tz \subseteq S$ for some $z \in \mathrm{Z}(G) \cap B$. In particular $(T \cap A)z \subseteq S \cap A = B$. Since $S \cap A \subseteq T \cap A$ and $S \cap A$ is a maximal order, it follows that $T \cap A = S \cap A$. Because $S \subseteq T$, the assumption on S then implies that $T = S$. Therefore S is a maximal order. ∎

In order to illustrate the above proposition with a concrete example, we start with the following construction of a monoid that contains the abelian monoid generated by a_1, a_2, a_3, a_4 and defined by the extra relation $a_1 a_2 = a_3 a_4$. It was shown in [2] that the latter is a cancellative monoid that is a maximal order.

Example 3.2 *The abelian monoid $B = \langle a_1, a_2, a_3, a_4, a_5, a_6 \rangle$ defined by the relations $a_1 a_2 = a_3 a_4 = a_5 a_6$ is a cancellative monoid that is a maximal order (in its torsion free group of quotients).*

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Proof. Let $F = \langle x_1, \ldots, x_8 \rangle$ be a free abelian monoid of rank 8. Define

$$b_1 = x_1 x_2 x_3 x_4, \ b_2 = x_5 x_6 x_7 x_8, \ b_3 = x_1 x_2 x_5 x_6,$$

$$b_4 = x_3 x_4 x_7 x_8, \ b_5 = x_1 x_3 x_5 x_7, \ b_6 = x_2 x_4 x_6 x_8.$$

Clearly, these b_i satisfy the defining relations for B. We claim that actually $B \cong \langle b_1, \ldots, b_6 \rangle$ under the map determined by $a_i \mapsto b_i$, $i = 1, \ldots, 6$. In order to prove this, suppose that there is a relation $w = v$, where w, v are nontrivial words in b_i. We need to show that this relation follows from $b_1 b_2 = b_3 b_4 = b_5 b_6$. Cancelling in F, if needed, we may assume that each b_i appears at most on one side of the relation. We also may assume that not both sides are divisible in $\langle b_1, \ldots, b_6 \rangle$ by one of the equal words $b_1 b_2$, $b_3 b_4$ and $b_5 b_6$. Further, on both sides of $v = w$ we need some b_i with an even i. Indeed, suppose the contrary, then x_8 is not involved in v and w. Hence also b_5 cannot occur because of x_7. But, as b_1 and b_3 generate a free abelian monoid of rank 2, it then follows that v and w are identical words in the b_i's, as desired. Hence, by symmetry we may assume that w contains $b_2^{i_2}$ with $i_2 > 0$ and w does not contain b_4 nor b_6 as a factor, and v does not contain b_2 as a factor and contains $b_4^{i_4} b_6^{i_6}$ for some nonnegative i_4, i_6 with $i_2 = i_4 + i_6$ (the latter follows by taking into the account the degree of x_8 in the respective words). Looking at x_4 we then get that b_1 appears in w. If $i_6 = 0$ then (looking at x_6) we get that b_3 is in v, a contradiction because $b_1 b_2 = b_3 b_4$ divides then both v and w. So $i_6 > 0$. Also v must contain x_1, x_5 and so b_3 or b_5 is in v. The latter is not possible because then v and w are divisible by $b_1 b_2 = b_5 b_6$, a contradiction. Thus, b_3 occurs in v. Then we must have $i_4 = 0$ because otherwise $b_1 b_2 = b_3 b_4$ divides v and w. So $w = b_1^{i_1} b_2^{i_2} b_5^{i_5} = b_3^{i_3} b_6^{i_6} = v$ for some $i_1, i_2, i_6 > 0$ and $i_3, i_5 \geq 0$. Then the exponents of x_7 show that $i_2 = i_5 = 0$, a contradiction. The claim follows. Hence we may indeed identify a_i with b_i and B with $\langle b_1, \ldots, b_6 \rangle$.

Next, suppose that $w = a_2^{i_2} a_3^{i_3} a_4^{i_4} a_5^{i_5} \in F \cap \mathrm{gr}(B)$. This is equivalent to the conditions: $i_3 + i_5 \geq 0, i_3 \geq 0, i_4 + i_5 \geq 0, i_4 \geq 0, i_2 + i_3 + i_5 \geq 0, i_2 + i_3 \geq 0, i_2 + i_4 + i_5 \geq 0, i_2 + i_4 \geq 0$. Let $j = \min\{i_3, i_4\} \geq 0$. Then $j + i_2 + i_5 \geq 0$. We choose $s, t \geq 0$ so that $i_2 + s, i_5 + t \geq 0$ and $s + t = j$. Then $w = a_1^s a_2^{i_2+s} a_3^{i_3-j} a_4^{i_4-j} a_5^{i_5+t} a_6^t$. It is thus clear that $w \in B$. So $B = F \cap \mathrm{gr}(B)$. Since F is a maximal order it then easily follows that B is a maximal order. ∎

We conclude with the promised illustration of Theorem 2.4. This example also shows that Theorem 1.1 cannot be extended to prime ideals of height exceeding 1.

Example 3.3 *Let K be any field and let $R = K\langle x_1, x_2, x_3, x_4 \rangle$ be the algebra defined by the following relations:*

$$x_1 x_4 = x_2 x_3, x_1 x_3 = x_2 x_4, x_3 x_1 = x_4 x_2$$

$$x_3 x_2 = x_4 x_1, x_1 x_2 = x_3 x_4, x_2 x_1 = x_4 x_3.$$

Clearly, $R = K[S]$ for the monoid S defined by the same presentation. Then S is cancellative (but the group SS^{-1} is not torsion free) and R is a prime

Noetherian PI-algebra that is a maximal order. Furthermore, there exists a prime ideal P of S so that $K[P]$ is not a prime ideal of $K[S]$.

Proof. Notice that each of the permutations $(12)(34), (13)(24)$ and $(14)(23)$ determines an automorphism of S. First we list some equalities in S, namely all relations between the elements of length 3. For brevity, we use the index i in place of the generator x_i.

$$112 = 134 = 244, \quad 113 = 124 = 344, \quad 114 = 123 = 343 = 321 = 411$$
$$221 = 243 = 133, \quad 224 = 213 = 433, \quad 223 = 214 = 434 = 412 = 322$$
$$331 = 342 = 122, \quad 334 = 312 = 422, \quad 332 = 341 = 121 = 143 = 233$$
$$442 = 431 = 211, \quad 443 = 421 = 311, \quad 441 = 432 = 212 = 234 = 144$$

$$131 = 142 = 232 = 241, \quad 141 = 132 = 242 = 231$$
$$313 = 423 = 414 = 324, \quad 323 = 314 = 424 = 413.$$

It follows easily that $A = \langle x_1^2, x_2^2, x_3^2, x_4^2 \rangle$ is an abelian submonoid and it is normal, that is $xA = Ax$ for every $x \in S$. Moreover $x_1^2 x_4^2 = x_2^2 x_3^2$ because $x_2 x_2 x_3 x_3 = x_2 x_3 x_3 x_2 = x_1 x_4 x_4 x_1 = x_4 x_4 x_1 x_1$.

Let $a_1 = x_1 x_4, a_2 = x_4 x_1, a_3 = x_1^2, a_4 = x_4^2, a_5 = x_2^2, a_6 = x_3^2$ and $B = \langle a_1, a_2, a_3, a_4, a_5, a_6 \rangle$. From the above equalities it follows that B is abelian and $sB = Bs$ for every $s \in S$.

Every element of S that is a word of length 3 is either of the form xyy or of the form $xx_1 x_4, xx_4 x_1$ for some x. It is also easy to see that every element of S is of the form $zu_1 w_1 \cdots u_r w_r$ or $zu_1 w_1 \cdots u_r w_r u_{r+1}$ or zc, where $z \in A$ and $c \in S$ is an element of length at most 2 in the x_j and either $u_i \in \{x_1, x_2\}, w_i \in \{x_3, x_4\}$ or $u_i \in \{x_3, x_4\}, w_i \in \{x_1, x_2\}$ (for all i). So we may assume that $c \in \{x_1 x_2, x_2 x_1\}$. Here r is a non-negative integer. Using the relations listed above (especially $x_1 x_4 x_1 = x_2 x_4 x_2, x_4 x_1 x_4 = x_3 x_1 x_3$) it may be checked that the even powers of $x_1 x_4 = x_2 x_3$ and $x_2 x_4 = x_1 x_3$ are equal and the even powers of $x_4 x_1 = x_3 x_2$ and $x_3 x_1 = x_4 x_2$ are equal. Also $\{x_4 x_1, x_1 x_4\}\{x_1 x_2, x_2 x_1\} \subseteq A\{x_1 x_3, x_3 x_1\}$. It follows that possible forms of elements of S are

$$z(x_1 x_4)^i, \ z(x_1 x_4)^i x_1, \ z(x_1 x_4)^i x_2, \ z(x_1 x_4)^i x_1 x_3, \ z x_1 x_2 \tag{2}$$
$$z(x_4 x_1)^i, \ z(x_4 x_1)^i x_4, \ z(x_4 x_1)^i x_3, \ z(x_4 x_1)^i x_3 x_1, \ z x_2 x_1, \tag{3}$$

with $z \in A, i \geq 0$. This leads to

$$S = B \cup Bx_1 \cup Bx_2 \cup Bx_3 \cup Bx_4 \cup Bx_1 x_3 \cup Bx_3 x_1 \cup Bx_1 x_2 \cup Bx_2 x_1. \tag{4}$$

It follows that $K[S]$ is finite module over the finitely generated commutative algebra $K[B]$. Hence $K[S]$ is a Noetherian PI-algebra.

Let C be the free abelian group of rank 4 generated by elements a, b, c, d. Let X be the free monoid on x_1, x_2, x_3, x_4. Consider the monoid homomorphism

$\phi : X \longrightarrow M_4(K[C])$ defined by

$$
x_1 \mapsto \begin{pmatrix} 0 & a & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & a^{-1}bc \\ 0 & 0 & 1 & 0 \end{pmatrix}, \qquad
x_2 \mapsto \begin{pmatrix} 0 & 0 & b & 0 \\ 0 & 0 & 0 & a^{-1}bc \\ 1 & 0 & 0 & 0 \\ 0 & ab^{-1} & 0 & 0 \end{pmatrix},
$$

$$
x_3 \mapsto \begin{pmatrix} 0 & 0 & bcd^{-1} & 0 \\ 0 & 0 & 0 & a^{-1}bd \\ b^{-1}d & 0 & 0 & 0 \\ 0 & ad^{-1} & 0 & 0 \end{pmatrix}, \qquad
x_4 \mapsto \begin{pmatrix} 0 & bcd^{-1} & 0 & 0 \\ a^{-1}d & 0 & 0 & 0 \\ 0 & 0 & 0 & d \\ 0 & 0 & ad^{-1} & 0 \end{pmatrix}.
$$

It is easy to check that these matrices satisfy the defining relations of S, so ϕ can be viewed as a homomorphism from S to the group of monomial matrices over C. Moreover

$$
a_3 \mapsto \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & a^{-1}bc & 0 \\ 0 & 0 & 0 & a^{-1}bc \end{pmatrix}, \qquad
a_5 \mapsto \begin{pmatrix} b & 0 & 0 & 0 \\ 0 & c & 0 & 0 \\ 0 & 0 & b & 0 \\ 0 & 0 & 0 & c \end{pmatrix}
$$

$$
a_4 \mapsto \begin{pmatrix} a^{-1}bc & 0 & 0 & 0 \\ 0 & a^{-1}bc & 0 & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & a \end{pmatrix}, \qquad
a_6 \mapsto \begin{pmatrix} c & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & b \end{pmatrix},
$$

$$
a_1 \mapsto \begin{pmatrix} d & 0 & 0 & 0 \\ 0 & bcd^{-1} & 0 & 0 \\ 0 & 0 & bcd^{-1} & 0 \\ 0 & 0 & 0 & d \end{pmatrix}, \qquad
a_2 \mapsto \begin{pmatrix} bcd^{-1} & 0 & 0 & 0 \\ 0 & d & 0 & 0 \\ 0 & 0 & d & 0 \\ 0 & 0 & 0 & bcd^{-1} \end{pmatrix}.
$$

The projection of the group C' generated by $\phi(a_1), \phi(a_3), \phi(a_5), \phi(a_6)$ onto the $(1,1)$-entry contains a, b, c, d, whence it is free abelian of rank 4. So $C' \cong C$. In particular, ϕ is injective on $B_0 = \langle a_1, a_3, a_5, a_6 \rangle$, and thus B_0 is a free abelian monoid of rank 4. Let B' be the abelian monoid with presentation $\langle y_1, y_2, y_3, y_4, y_5, y_6 \mid y_1 y_2 = y_3 y_4 = y_5 y_6 \rangle$. Clearly, we have natural homomorphisms $B' \longrightarrow B \longrightarrow \phi(B)$. We also know that $\phi(B)$ generates a free abelian group of rank ≥ 4 and B' has a group of quotients that is free abelian of rank 4. So B' and $\phi(B)$ must be isomorphic, since otherwise under the map $B' \longrightarrow \phi(B)$ we have to factor out an additional relation and the rank would decrease. It follows that ϕ is injective on B and thus, because of Example 3.2, B is a cancellative maximal order with group of quotients $N = BB^{-1} \cong \mathbb{Z}^4$. Note that $AA^{-1} = \mathrm{gr}(a_3, a_5, a_6)$. Using the defining relations $a_1 a_2 = a_3 a_4 = a_5 a_6$ for B, it is readily verified that if $i, j, k \in \mathbb{Z}$ then $a_3^i a_5^j a_6^k \in A = \langle a_3, a_4, a_5, a_6 \rangle$ if and only if $j, k \geq 0$ and $\min(j, k) + i \geq 0$.

The images under ϕ of the first 4 types listed in (2) have different patterns of nonzero entries in $M_4(K[C])$. The same applies to the first 4 types in (3). Notice that ϕ is injective on each of the 10 types. Suppose that $s =$

$z(x_1x_4)^ix_1, t = z'(x_4x_1)^jx_4$ have the same image. Then $sx_4 = z(x_1x_4)^{i+1}$ and $tx_4 = z'(x_4x_1)^jx_4^2$ have equal images. This is not possible because $(x_1x_4)^{i+1}A \cap (x_4x_1)^jA = \emptyset$ by the above description of the group $BB^{-1} \cong \phi(B)\phi(B)^{-1}$. Similarly one deals with elements of any two different types listed in (2) and (3), showing that only elements the form $s = zx_1x_2, t = z'x_2x_1$, where $z, z' \in A$, can have equal images. Then $\phi(zx_1x_1x_4x_1) = \phi(tx_4x_2) = \phi(sx_4x_2) = \phi(z'x_4x_1x_2x_2)$ and cancellativity of $\phi(B) \cong B$ yields $zx_1x_1 = z'x_2x_2$. Write $z = a_3^i a_5^j a_6^k$, with $i, j, k \in \mathbb{Z}$. Since $z, z' \in A$, the above yields that $z' = a_3^{i+1} a_5^{j-1} a_6^k$ and $j, k \geq 0, -i$ and $j-1, k \geq 0, -(i+1)$. Notice that $a_3x_2x_1 = x_1x_1x_2x_1 = x_2x_2x_1x_2 = a_5x_1x_2$. Therefore, if $j-1 \geq -i$ then $a_3^i a_5^{j-1} a_6^k \in A$ and hence $zx_1x_2 = a_3^i a_5^{j-1} a_6^k a_5 x_1 x_2 = a_3^i a_5^{j-1} a_6^k a_3 x_2 x_1 = z'x_2x_1$. On the other hand, if $j = -i$ then $a_4^{j-1} a_6^{k-j} \in A$ and $a_4x_1x_2 = a_6x_2x_1$. Hence we also get $zx_1x_2 = a_4^j a_6^{k-j} x_1 x_2 = a_4^{j-1} a_6^{k-j} a_4 x_1 x_2 = a_4^{j-1} a_6^{k-j} a_6 x_2 x_1 = z'x_2x_1$.

It follows that ϕ is injective on all elements of types (2),(3). Therefore ϕ is an embedding and thus S is cancellative.

We identify S with $\phi(S)$. Put $G = SS^{-1}$. Then $G = N \cup x_1N \cup x_2N \cup x_1x_2N$. Moreover $N \cong \mathbb{Z}^4$ and N is a normal subgroup with G/N the four group. From (4) it follows that $S \cap N = B$.

We now show that G is dihedral free. To prove this, notice that S acts by conjugation on B and the generators x_i of S correspond to the following permutations σ_i of the generating set of B (the numbers $1, 2, 3, 4, 5, 6$ correspond to the generators $a_1, a_2, a_3, a_4, a_5, a_6$).

$$\sigma_1 = (12)(56), \ \sigma_2 = (12)(34), \ \sigma_3 = (12)(34), \ \sigma_4 = (12)(56). \tag{5}$$

Suppose $D \subseteq G$ is an infinite dihedral group such that the normalizer $N_G(D)$ of D in G is of finite index. Let $t \in D$ be an element of order 2. Then there exists $k \geq 1$ such that $a_i^k t a_i^{-k} t \in D$ for $i = 1, 2, \ldots, 6$. Clearly

$$a_i^k t a_i^{-k} t = a_i^k a_{\sigma(i)}^k t^2 = a_i^k a_{\sigma(i)}^k,$$

where σ is the automorphism of N determined by t. Since $t \in Nx_1 \cup Nx_2 \cup Nx_1x_2$, it follows that σ is determined by the conjugation by x_1, x_2 or x_1x_2. So σ permutes exactly two of the pairs $a_1, a_2; a_3, a_4$ and a_5, a_6. It follows that $a_i^k a_j^k, a_p^k a_q^k \in D$ for two different pairs i, j and p, q. Hence $\mathrm{rk}(N \cap D) \geq 2$, a contradiction. Therefore G indeed is a dihedral free group.

Next we show that $\Delta^+(G)$ is trivial. For this, let F be a finite normal subgroup of G. Since N is torsion free, it is clear that F is isomorphic with a subgroup of $\mathbb{Z}_2 \times \mathbb{Z}_2$. A nontrivial element $t \in F$ must be of order 2, whence as above we get that $a_i^n a_j^n t \in F$ for some $i \neq j$ and every $n \geq 1$. Therefore F is infinite, a contradiction. It follows that $\Delta^+(G)$ is trivial. Therefore $K[G]$ is prime and hence $K[S]$ also is prime.

Note that in G we have $x_2^{-1}x_1 = x_3x_4^{-1} = x_4x_3^{-1}$ and $x_1x_2^{-1} = x_3^{-1}x_4 = x_4^{-1}x_3$. So $x_2^{-1}x_1$ is an element of order 2.

We describe the minimal prime ideals of S. For this we first notice that it

is easy to see that the minimal prime ideals of B are:

$$Q = Q_1 = (a_1, a_3, a_5), \qquad Q_2 = Q^{x_1} = (a_2, a_3, a_6),$$
$$Q_3 = Q^{x_2} = (a_2, a_4, a_5), \qquad Q_4 = Q^{x_1 x_2} = (a_1, a_4, a_6),$$
$$Q' = Q_5 = (a_2, a_3, a_5), \qquad Q_6 = (Q')^{x_1} = (a_1, a_3, a_6),$$
$$Q_7 = (Q')^{x_2} = (a_1, a_4, a_5), \qquad Q_8 = (Q')^{x_1 x_2} = (a_2, a_4, a_6).$$

Because of (5), it is easily verified that $a_1 a_2$ is a central element of S and that every ideal of S contains a positive power of $a_1 a_2$. In particular, $a_1 a_2$ belongs to every prime ideal of S. Consider in B the following G-invariant ideals:

$$M = (a_1 a_3 a_5, a_2 a_3 a_6, a_2 a_4 a_5, a_1 a_4 a_6, a_1 a_2)$$

and

$$M' = (a_2 a_3 a_5, a_1 a_4 a_5, a_1 a_3 a_6, a_2 a_4 a_6, a_1 a_2).$$

Again because of (5), it is easy to see that $bSb' \subseteq a_1 a_2 S$ for every defining generator b of M and b' of M'. It follows that a prime ideal of S contains M or M'.

Notice that $x a_1 a_3 a_5 y \in xy\{a_1 a_3 a_5, a_2 a_4 a_5, a_2 a_3 a_6, a_1 a_4 a_6\}$ for every $x, y \in S$. Therefore $S a_1 a_3 a_5 S \cap \langle a_2, a_3, a_5 \rangle = \emptyset$ (for example, $xy a_1 a_3 a_5 \notin \langle a_2, a_3, a_5 \rangle$ because otherwise $xy = a_1^{-1} a_3^{-1} a_5^{-1} \langle a_2, a_3, a_5 \rangle \cap (S \cap N)$, which is not possible because $S \cap N = B$ and $N = \mathrm{gr}(a_1, a_2, a_3, a_5)$ is free abelian of rank 4). So there exists a (unique) ideal P of S that is maximal with respect to the property $P \cap \langle a_2, a_3, a_5 \rangle = \emptyset$. It is easy to see that P is a prime ideal of S. Since $a_2 a_3 a_5 \notin P$, we get that $M' \not\subseteq P$, whence $M \subseteq P$.

Let $E = a_2 a_3 a_5 \langle a_2, a_3, a_5 \rangle$. Then

$$x_4 \langle a_1, a_3, a_6 \rangle x_1 \subseteq E, \; x_2 \langle a_1, a_4, a_5 \rangle x_2 \subseteq E, \; x_1 x_3 \langle a_2, a_4, a_6 \rangle x_3 x_1 \subseteq E. \qquad (6)$$

Therefore $P \cap \langle a_1, a_3, a_6 \rangle = \emptyset, P \cap \langle a_1, a_4, a_5 \rangle = \emptyset, P \cap \langle a_2, a_4, a_6 \rangle = \emptyset$. For every element $b \in B \setminus M$ the ideal bB intersects one of the sets

$$a_2 a_3 a_5 \langle a_2, a_3, a_5 \rangle, a_1 a_3 a_6 \langle a_1, a_3, a_6 \rangle, a_1 a_4 a_5 \langle a_1, a_4, a_5 \rangle, a_2 a_4 a_6 \langle a_2, a_4, a_6 \rangle.$$

Since these sets do not intersect P, we get that $P \cap B = M$. Hence it is G-invariant and, because $M \subseteq Q_1 \cap Q_2 \cap Q_3 \cap Q_4$ and all Q_i are minimal primes of B, Corollary 1.3 yields that $P \cap B = M = Q_1 \cap Q_2 \cap Q_3 \cap Q_4 = \mathcal{B}(SMS)$ and P is a minimal prime ideal of S. A similar argument shows that there exists an ideal P' of S that is maximal with respect to the property $P' \cap \langle a_1, a_3, a_5 \rangle = \emptyset$, and it follows that $P' \cap B = M' = Q_5 \cap Q_6 \cap Q_7 \cap Q_8 = \mathcal{B}(SM'S)$ is the only other minimal prime of S.

In order to continue the proof we first show the following claim on the representation of elements of B.

Claim: Presentation
An element $b = a_1^{\alpha_1} a_3^{\alpha_3} a_4^{\alpha_4} a_6^{\alpha_6}$ of $N = BB^{-1}$ is in B (with each $\alpha_i \in \mathbb{Z}$) if and only if $\alpha_3, \alpha_4 \geq 0$ and either (i) $\alpha_1 \geq 0$ and $\min(\alpha_3, \alpha_4) + \alpha_6 \geq 0$ or (ii)

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$\alpha_1 < 0$, $\alpha_3 + \alpha_1, \alpha_4 + \alpha_1 \geq 0$ and $\min(\alpha_3, \alpha_4) + \alpha_1 + \alpha_6 \geq 0$, or equivalently, $\min(\alpha_3 + \alpha_1, \alpha_4 + \alpha_1) \geq \max(0, -\alpha_6)$.

That the first condition is sufficient is easily verified. For the second condition, we rewrite $a_1^{\alpha_1}$ as $a_3^{\alpha_1} a_4^{\alpha_1} a_2^{-\alpha_1}$ and the result follows because of the first part (by interchanging a_1 with a_2).

To prove that they are necessary, suppose $b = a_1^{\alpha_1} a_3^{\alpha_3} a_4^{\alpha_4} a_6^{\alpha_6} \in B$, with each $\alpha_i \in \mathbb{Z}$. Because $Ba_4 \cap \langle a_1, a_3, a_6 \rangle = \emptyset$ and $Ba_3 \cap \langle a_1, a_4, a_6 \rangle = \emptyset$, it follows that $\alpha_3, \alpha_4 \geq 0$. Suppose now that $\alpha_1 \geq 0$. Then

$$b = a_1^{\alpha_1} a_3^{\alpha_3 - \min(\alpha_3, \alpha_4)} a_4^{\alpha_4 - \min(\alpha_3, \alpha_4)} a_5^{\min(\alpha_3, \alpha_4)} a_6^{\alpha_6 + \min(\alpha_3, \alpha_4)}$$

and thus $a_1^{\alpha_1} a_3^{\alpha_3 - \min(\alpha_3, \alpha_4)} a_4^{\alpha_4 - \min(\alpha_3, \alpha_4)} a_5^{\min(\alpha_3, \alpha_4)} \in Ba_6^{-(\alpha_6 + \min(\alpha_3, \alpha_4))}$. Since the exponent of a_3 or a_4 is 0, this implies that $\alpha_6 + \min(\alpha_3, \alpha_4) \geq 0$, as desired.

On the other hand, suppose that $\alpha_1 < 0$. Then $b = a_2^{-\alpha_1} a_3^{\alpha_3 + \alpha_1} a_4^{\alpha_4 + \alpha_1} a_6^{\alpha_6}$. The previous case yields that $\alpha_3 + \alpha_1 \geq 0$, $\alpha_4 + \alpha_1 \geq 0$ and $\min(\alpha_3 + \alpha_1, \alpha_4 + \alpha_1) + \alpha_6 \geq 0$, again as desired. This proves the claim *Presentation*.

Next we show that S is a maximal order. First note that B is a maximal order by Example 3.2. So, because of Proposition 3.1, it is sufficient to prove that if $s \in SS^{-1} \setminus S$ then $\langle S, s \rangle \cap N$ strictly contains B.

We know that $G = SS^{-1} = N \cup x_1 N \cup x_2 N \cup x_1 x_2 N$. If $s \in N \setminus B$ then clearly $\langle S, s \rangle \cap N$ strictly contains B. So, there are three cases to be dealt with: (1) $s = bx_1 \in G \setminus S$, (2) $s = bx_2 \in G \setminus S$, and (3) $s = bx_1 x_2 \in G \setminus S$, where $b \in N \setminus B$.

Case (1): $s = bx_1 \in G \setminus S$. Obviously, $bx_1 x_4 = ba_1 \in \langle S, s \rangle$. If $ba_1 \in N \setminus B$, then we are done. So suppose $ba_1 \in B$ and $b \in N \setminus B$. We write b in the following form:

$$a_1^{\alpha_1} a_3^{\alpha_3} a_4^{\alpha_4} a_6^{\alpha_6}.$$

First, suppose that $\alpha_1 \geq 0$. By the claim *Presentation*, from $ba_1 \in B$ we get that $\min(\alpha_3, \alpha_4) \geq \max(-\alpha_6, 0)$. But, also from $b \in N \setminus B$ we get that $\min(\alpha_3, \alpha_4) < \max(-\alpha_6, 0)$, a contradiction. So, α_1 has to be strictly negative.

Therefore, assume that $\alpha_1 < 0$. Then

$$b = a_1^{\alpha_1} a_3^{\alpha_3} a_4^{\alpha_4} a_6^{\alpha_6} = a_2^{-\alpha_1} a_3^{\alpha_3 + \alpha_1} a_4^{\alpha_3 + \alpha_1} a_6^{\alpha_6}$$

and

$$ba_1 = a_2^{-\alpha_1 - 1} a_3^{\alpha_3 + \alpha_1 + 1} a_4^{\alpha_3 + \alpha_1 + 1} a_6^{\alpha_6}.$$

By the claim *Presentation* (by interchanging a_1 and a_2), we get that $\min(\alpha_3 + \alpha_1 + 1, \alpha_4 + \alpha_1 + 1) \geq \max(-\alpha_6, 0)$, but also $\min(\alpha_3 + \alpha_1, \alpha_4 + \alpha_1) < \max(-\alpha_6, 0)$. Hence, $\min(\alpha_3 + \alpha_1 + 1, \alpha_4 + \alpha_1 + 1) = \max(-\alpha_6, 0)$. Suppose $\alpha_6 \geq 0$. Then $\min(\alpha_3 + \alpha_1 + 1, \alpha_4 + \alpha_1 + 1) = 0$. Therefore, $b = a_1^{\alpha_1} a_3^{\alpha_3} a_4^{\alpha_4} a_6^{\alpha_6} = a_1^{-1} a_2^{\alpha_3} a_4^{\alpha_4 - \alpha_3} a_6^{\alpha_6}$ and $\alpha_4 > \alpha_3$ or $b = a_1^{-1} a_2^{\alpha_4} a_3^{\alpha_3 - \alpha_4} a_6^{\alpha_6}$ and $\alpha_3 \geq \alpha_4$. In the first case we get that $bx_1 \in S$, since $x_4^{-1} x_1^{-1} x_4 x_4 x_1 = x_4$. So this case gives a contradiction and is hence impossible. In the second case, $bx_1 x_1 = a_1^{-1} a_2^{\alpha_4} a_3^{\alpha_3 - \alpha_4 + 1} a_6^{\alpha_6} \in (\langle S, s \rangle \cap N) \setminus B$, as desired. If $\alpha_6 < 0$, $\min(\alpha_3 + \alpha_1 + 1, \alpha_4 + \alpha_1 + 1) = -\alpha_6$ and therefore $b = a_1^{\alpha_1} a_3^{\alpha_3} a_4^{\alpha_4} a_6^{\alpha_6} = a_1^{-1} a_2^{-\alpha_1 - 1} a_3^{\alpha_3 - \alpha_4} a_5^{-\alpha_6}$

if $\alpha_4 \leq \alpha_3$ or $b = a_1^{-1}a_2^{-\alpha_1-1}a_4^{\alpha_4-\alpha_3}a_5^{-\alpha_6}$ if $\alpha_3 < \alpha_4$. Since $-\alpha_1 - 1 \geq 0$ and $-\alpha_6 > 0$, by interchanging a_5 and a_6, this case is completely similar to the case where $\alpha_6 \geq 0$. This finishes the proof of Case (1).

Case (2): $s = bx_2 \in G \setminus S$. The permutation $\sigma = (12)(34)$ determines an automorphism σ on S with $\sigma(B) = B$. Clearly, $\sigma(s) = b'x_1$ with $b' = \sigma(b) \notin B$. From Case (1) we get that $\langle S, \sigma(s) \rangle \cap N$ properly contains B. Again applying σ to the latter we get that $\langle S, s \rangle \cap N$ properly contains B, as required.

Case (3): $s = bx_1x_2 \in G \setminus S$. Clearly, $bx_1x_2x_2x_4 = ba_1a_6 \in \langle S, s \rangle$. If $ba_1a_6 \in N \setminus B$, then we are done. So suppose $ba_1a_6 \in B$ and $b \in N \setminus B$. Write $b = a_1^{\alpha_1}a_3^{\alpha_3}a_4^{\alpha_4}a_6^{\alpha_6}$. We know that $\alpha_3, \alpha_4 \geq 0$ and we consider again the two cases: $\alpha_1 \geq 0$ and $\alpha_1 < 0$, separately.

First assume that $\alpha_1 \geq 0$. Since $ba_1a_6 \in B$ and $b \notin B$, we get that $\min(\alpha_3, \alpha_4) = -\alpha_6 - 1$. Then $b = a_1^{\alpha_1}a_3^{\alpha_3}a_4^{\alpha_4}a_6^{\alpha_6} = a_1^{\alpha_1}a_3^{\alpha_3-\alpha_4}a_5^{-\alpha_6-1}a_6^{-1}$, if $\alpha_4 \leq \alpha_3$, or $b = a_1^{\alpha_1}a_4^{\alpha_4-\alpha_3}a_5^{-\alpha_6-1}a_6^{-1}$, if $\alpha_3 < \alpha_4$. Now, let $\alpha_1 > 0$. Then $bx_1x_2 \in S$, since $x_1x_4x_3^{-1}x_3^{-1}x_1x_2 = x_2x_4$, a contradiction. So this case is impossible. Therefore, $\alpha_1 = 0$. If $\alpha_3 < \alpha_4$, then $bx_1x_2 \in S$, since $x_4x_4x_3^{-1}x_3^{-1}x_3x_4 = x_2x_1$, a contradiction, so this case is again impossible. If $\alpha_4 \leq \alpha_3$, then $bx_1x_2x_3x_1 = ba_2a_3 \in (\langle S, s \rangle \cap N) \setminus B$, as desired.

Finally, suppose $\alpha_1 < 0$. Then $b = a_1^{\alpha_1}a_3^{\alpha_3}a_4^{\alpha_4}a_6^{\alpha_6} = a_2^{-\alpha_1}a_3^{\alpha_3+\alpha_1}a_4^{\alpha_4+\alpha_1}a_6^{\alpha_6}$ and $ba_1a_6 = a_2^{-\alpha_1-1}a_3^{\alpha_3+\alpha_1+1}a_4^{\alpha_4+\alpha_1+1}a_6^{\alpha_6+1}$. By the claim *Presentation* (by interchanging a_1 and a_2), we get that $\min(\alpha_3+\alpha_1+1, \alpha_4+\alpha_1+1) \geq \max(-\alpha_6 - 1, 0)$, but also $\min(\alpha_3 + \alpha_1, \alpha_4 + \alpha_1) < \max(-\alpha_6, 0)$. If $\alpha_6 \geq 0$, $\min(\alpha_3, \alpha_4) + \alpha_1 = -1$ and $b = a_1^{-1}a_2^{\alpha_4}a_3^{\alpha_3-\alpha_4}a_6^{\alpha_6}$, if $\alpha_4 \leq \alpha_3$, or $b = a_1^{-1}a_2^{\alpha_3}a_4^{\alpha_4-\alpha_3}a_6^{\alpha_6}$, if $\alpha_3 < \alpha_4$. So, we have the same conditions here as in Case 1 and the result follows. If $\alpha_6 < 0$, there are two possibilities: $\min(\alpha_3, \alpha_4) + \alpha_1 = -\alpha_6 - 1$ or $\min(\alpha_3, \alpha_4) + \alpha_1 = -\alpha_6 - 2$. In the first case, we get that $b = a_1^{-1}a_2^{-\alpha_1-1}a_3^{\alpha_3-\alpha_4}a_5^{-\alpha_6}$ if $\alpha_4 \leq \alpha_3$, or $b = a_1^{-1}a_2^{-\alpha_1-1}a_4^{\alpha_4-\alpha_3}a_5^{-\alpha_6}$ if $\alpha_3 < \alpha_4$. Since $-\alpha_1 - 1 \geq 0$ and $-\alpha_6 > 0$, by interchanging a_5 and a_6, this case is completely similar to the case where $\alpha_6 \geq 0$ and hence can also be treated as in Case 1. Finally, if the second possibility holds, that is $\min(\alpha_3, \alpha_4) + \alpha_1 = -\alpha_6 - 2$, then $b = a_1^{-2}a_2^{-2-\alpha_1}a_3^{\alpha_3-\alpha_4}a_5^{-\alpha_6}$, if $\alpha_4 \leq \alpha_3$, or $b = a_1^{-2}a_2^{-2-\alpha_1}a_4^{\alpha_4-\alpha_3}a_5^{-\alpha_6}$, if $\alpha_4 > \alpha_3$, so we always get that $ba_2a_3 \in (\langle S, s \rangle \cap N) \setminus B$, as desired.

This finishes the proof of the fact that S is a maximal order. Since $P \cap B$ is invariant for every minimal prime P of S, it then follows from Theorem 2.4 that $K[S]$ is a maximal order.

Let V be the ideal of S generated by the elements $a_3, a_4, a_5, a_6, x_1x_2, x_2x_1$. It easily follows that the elements of $S \setminus V$ are of the form

$$(x_1x_4)^i, \ (x_1x_4)^ix_1, \ (x_1x_4)^ix_2, \ (x_1x_4)^ix_1x_3,$$
$$(x_4x_1)^i, \ (x_4x_1)^ix_4, \ (x_4x_1)^ix_3, \ (x_4x_1)^ix_3x_1,$$

where i is a non-negative integer. Then $S \setminus V = \{1\} \cup I$ where the set I can be written in matrix format as a union of disjoint sets:

$$\begin{pmatrix} I_{11} & I_{12} \\ I_{21} & I_{22} \end{pmatrix},$$

with $I_{11} = \langle a_2 \rangle a_2 \cup \langle a_2 \rangle x_3 x_1$, $I_{12} = \langle a_2 \rangle x_3 \cup \langle a_2 \rangle x_4$, $I_{21} = x_1 \langle a_2 \rangle \cup x_2 \langle a_2 \rangle$ and $I_{22} = \langle a_1 \rangle a_1 \cup \langle a_1 \rangle x_1 x_3$. Moreover, $I_{ij} I_{kl} \subseteq I_{il}$ if $j = k$, and is contained in V otherwise. In S/V the set $I \cup \{0\}$ is an ideal and the semigroup I_{11} (treated as a subsemigroup of S) has a group of quotients $H = \mathrm{gr}(a_2, x_3 x_1)$. Since $a_2^2 = (x_3 x_1)^2$ and $a_2(x_3 x_1) = (x_3 x_1)a_2$, we get that H is isomorphic with $\mathbb{Z} \times \mathbb{Z}_2$. From the matrix pattern of I it follows that S/V is a prime semigroup. So V is a prime ideal of S. However, because $K[H]$ and thus $K[I_{11}]$ is not prime, standard generalized matrix ring arguments yield that $K[S]/K[V]$ is not prime. ∎

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